Math-156 Calculus 1 Final Examination December 8, 2009

Do any 10 questions. Each question is worth 20 points.
Show sufficient work to justify your answers.

1. (a) Let

$$G(x) = \begin{cases} x & \text{if } x < -3 \\ -3 & \text{if } -3 \le x < 0 \\ 2x & \text{if } x \ge 0 \end{cases}$$

Find
(i) $\lim_{x \to 3} G(x)$ (ii) $\lim_{x \to -3} G(x)$ (iii) the set of points of discontinuity of $G(x)$.

(b) Find the limit if it exists.
(i) $\lim_{x \to 0^-} e^{1/x}$ (ii) $\lim_{x \to 0^+} e^{1/x}$

2. (a) Let $f(x) = x - 2 + \cos x$. Use the Intermediate Value Theorem to show that the equation $f(x) = 0$ has at least one solution for x in $[0, \pi]$.
Note : You do not need to find the solution.

(b) Use the Squeeze Theorem to find the limit of $f(\theta)$ as $\theta \to 0$ if we know that

$$1 - 2\theta^2 < f(\theta) < \frac{\sin\theta}{\theta}$$

for $0 < |\theta| < \frac{\pi}{2}$.

3. (a) Let $f(x) = \frac{1}{2-x}$.
Use the definition of derivative $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ to compute the derivative of f .

(b) Let

$$f(x) = \begin{cases} x^3 + \frac{1}{16} & \text{if } x < \frac{1}{2} \\ \frac{3}{4}x^2 & \text{if } x \ge \frac{1}{2} \end{cases}$$

Determine whether f is differentiable at $x = \frac{1}{2}$. If so, find the value of the derivative there.

4. Find $\frac{dy}{dx}$. Simplify your answers.

(a) $y = x^2 + 3x - \dfrac{2}{x\sqrt{x}}$ (b) $y = \sin(x^2)$

(c) $y = \dfrac{1}{\sin^{-1}\sqrt{x}}$ (d) $y = \ln\left(\dfrac{x^2 \sin x}{\sqrt{1+x}}\right)$

5. (a) Find $f'(0)$, where

(a) $f(x) = \left(x^3 + x + 1\right)^e$ (b) $f(x) = \ln(e^{2x} + e^{-x})$

(b) Use implicit differentiation to find $\frac{dy}{dx}$, where

$$\ln(xy) = e^x + e^y$$

6. Let

$$f(x) = e^x(x^2 - 3)$$

Find the intervals on which f is increasing, the intervals on which f is decreasing, the intervals on which f is concave up, and the intervals on which f is concave down.

7. (a) Let

$$f(x) = x^3(x-1)^2$$

Find all relative extrema of f.

(b) Find the absolute maximum and minimum of $f(x) = x^3 - 6x + 1$ on the interval $[-2, 0]$.

8. A cylindrical can is to have a volume of 1200 cm^3. The material for the side of the can costs 2 cents per cm^2 and the material for the top and bottom of the can costs 3 cents per cm^2. Find the radius of the base of the can which minimizes the cost.

9. Evaluate the integrals

$$\text{(a)} \quad \int \frac{x^3 + x}{x^2}\, dx \qquad \text{(b)} \quad \int (5 + \sin 3x)^4 \cos 3x\, dx$$

$$\text{(c)} \quad \int_{-2}^{5} |x + 1|\, dx \qquad \text{(d)} \quad \int \left[\frac{1}{2\sqrt{1 - x^2}} - \frac{3}{1 + x^2} \right] dx$$

10. Consider the region in the first quadrant bounded by the curve $y = 16 - x^2$.
(a) Find an upper estimate of the area of the region by dividing the interval [0, 4] into 4 subintervals of equal length and constructing rectangles over the intervals.
(b) Find the area of the region.

11. The position of a particle on a number line is given by $P(t) = t^2 - t + 5$.
(a) Find the time when its instantaneous velocity is equal to its average velocity during the 4 seconds, $0 \le t \le 4$.
(b) What does the mean-value theorem say for this particle?

12. (a) Let

$$F(x) = \int_2^x \sqrt{3t^2 + 1}\, dt\,.$$

Find

$$(i) \quad F(2) \qquad (ii) \quad F'(2) \qquad (iii) \quad F''(2)$$

(b) Find the average value of

$$f(x) = e^x + e^{-x}$$

over the interval $\left[\ln \frac{1}{2},\ \ln 2\right]$.