HOWARD UNIVERSITY
DEPARTMENT OF MATHEMATICS
SENIOR COMPREHENSIVE EXAMINATION
NOVEMBER 2, 2013

Name: ______________________________

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Signature: ______________________________

⇒ This exam consists of 10 questions. Answer all the questions. Each question is worth 10 points.

⇒ Show all your work as neatly and legibly as possible on the Bluebook provided. **No work, no credit.**

⇒ Good Luck!

Question	**Points**	**Out of**
1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
8		10
9		10
10		10
Total		100
GRADE (P or F)		

10 points 1. Evaluate the following limits:

(a) $\lim_{x\to 0^+} \frac{\tan x - x}{x - \sin x}$

This is a 0/0 type indeterminate form. Using L'Hopital's rule repeatedly, we have

$$\begin{aligned}
\lim_{x\to 0^+} \frac{\tan x - x}{x - \sin x} &= \lim_{x\to 0^+} \frac{\sec^2 x - 1}{1 - \cos x} \\
&= \lim_{x\to 0^+} \frac{\tan^2 x}{1 - \cos x} \\
&= \lim_{x\to 0^+} \frac{2\tan x \sec^2 x}{1 + \sin x} \\
&= \frac{0}{1} = 1
\end{aligned}$$

(b) $\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$

This is also a 0/0 indeterminate form and repeated use of L'Hopitals rule helps:

$$\begin{aligned}
\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{\sin x}\right) &= \lim_{x\to 0} \left(\frac{\sin x - x}{x \sin x}\right) \\
&= \lim_{x\to 0} \left(\frac{\cos x - 1}{\sin x + x\cos x}\right) \\
&= \lim_{x\to 0} \left(\frac{-\sin x}{2\cos x - \sin x}\right) \\
&= \frac{0}{2} = 0.
\end{aligned}$$

(c) $\lim_{x\to\infty} x(\sqrt{x^2+4} - x)$ This is also 0/0. This time we will rationalize:

$$\begin{aligned}
\lim_{x\to\infty} x(\sqrt{x^2+4} - x) &= \lim_{x\to\infty} x(\sqrt{x^2+4} - x)\frac{\sqrt{x^2+4} - x}{\sqrt{x^2+4} + x} \\
&= \lim_{x\to\infty} \frac{x(x^2 + 4 - x^2)}{\sqrt{x^2+4} + x} \\
&= \lim_{x\to\infty} \frac{4x}{\sqrt{x^2+4} + x} \\
&= \lim_{x\to\infty} \frac{4}{\sqrt{1 + \frac{4}{x^2}} + 1} = 2.
\end{aligned}$$

10 points 2. Evaluate the following integrals:

(a) $\int \sec^3 x\, dx$
First we use integration by parts with:

$$\begin{aligned} u &= \sec x & dv &= \sec^2 x \\ du &= \sec x \tan x & v &= \tan x \end{aligned}$$

to obtain

$$\begin{aligned} \int \sec^3 x dx = \int \sec x \sec^2 x dx &= \tan x \sec x - \int \sec x \tan^2 x dx \\ &= \tan x \sec x - \int \sec^3 x dx + \int \sec x dx \end{aligned}$$

from which we get

$$2 \int \sec^3 x dx = \tan x \sec x + \int \sec x dx$$

and thus

$$\int \sec^3 x dx = \frac{1}{2} \tan x \sec x - \frac{1}{2} \ln|\sec x + \tan x| + C$$

(b) $\int \frac{e^t}{e^{2t} + 3e^t + 2}\, dt$ First we use the substitution $u = e^t$ and we get

$$\int \frac{du}{u^2 + 3u + 2}$$

which we may integrate by partial fraction and obtain

$$\int \frac{du}{u^2 + 3u + 2} = -\ln|u + 2| + \ln|u + 1| + C$$

and thus

$$\int \frac{e^t}{e^{2t} + 3e^t + 2}\, dt = -\ln|e^t + 2| + \ln|e^t + 1| + C$$

(c) $\int x \sin^{-1} x\, dx$

First by using integration by parts with :

$$\begin{aligned} u &= \sin^{-1} x & dv &= dx \\ du &= \frac{1}{\sqrt{1 - x^2}} & v &= x \end{aligned}$$

we obtain that

$$\int \sin^{-1} x dx = x \sin^{-1} x + \sqrt{1 - x^2} + c$$

Again by using integration by parts with :

$$u = x \qquad dv = \sin^{-1} x dx$$
$$du = dx \qquad v = x \sin^{-1} x + \sqrt{1 - x^2}$$

we obtain that

$$\int x \sin^{-1} x dx = x \sin^{-1} x + x\sqrt{1-x^2} - \int x \sin^{-1} x dx - \int \sqrt{1-x^2} dx$$

which gives:

$$\int x \sin^{-1} x dx = \frac{1}{2}[x \sin^{-1} x + x\sqrt{1-x^2}] - \int \sqrt{1-x^2} dx$$

By using Trigonometric substitution we obtain:

$$\int \sqrt{1-x^2} dx = \frac{1}{2} \sin^{-1} x + \frac{x}{2\sqrt{x^2+1}}$$

Thus

$$\int x \sin^{-1} x dx = \frac{1}{2}[x \sin^{-1} x + x\sqrt{1-x^2}] - [\frac{1}{2} \sin^{-1} x + \frac{x}{2\sqrt{x^2+1}}] + C$$

(d) $\int_0^\pi \int_x^\pi \frac{\sin y}{y} \, dy \, dx$ (Hint: Reverse the order of integration first)
When we reverse the order of integration we obtain

$$\int_0^\pi \int_x^\pi \frac{\sin y}{y} \, dy \, dx = \int_0^\pi \int_0^y \frac{\sin y}{y} \, dx \, dy$$

which gives

$$\int_0^\pi \sin y dy = -\cos y|_0^\pi = 2.$$

10 points 3. (a) State the Intermediate Value Theorem.

If f is a continuous function on the closed interval $[a, b]$ and $f(a) \leq y \leq f(b)$ or $f(b) \leq y \leq f(a)$ there exists a number $c \in [a, b]$ such that $f(c) = y$.

(b) Let $f(x)$ be a continuous function from $[0, 1]$ onto $[0, 1]$. Prove that there exists a c in $[0, 1]$ such that $f(c) = c$. (Hint: Use the Intermediate Value Theorem)

Proof: Consider the function $g(x) = f(x) - x$ which is obviously continuous since f is continuous. Then $g(0) = f(0) \geq 0$ and $g(1) = f(1) - 1 \leq 0$ since $f(1) \in [0, 1]$. Therefore, by intermediate value theorem there exists a number c in $[0, 1]$ such that $g(c) = 0$, which means that $f(c) = c$.

10 points 4. For the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 6 \end{bmatrix}$,

(a) Find all the eigenvalues of the matrix A.
Solution: The eigenvalues are solutions of the equation

$$det(A - \lambda I) = det \begin{bmatrix} 1-\lambda & 4 \\ 2 & 6-\lambda \end{bmatrix} = 0$$

This gives the equation:

$$(1-\lambda)(6-\lambda) - 8 = 0$$

Expanding and solving we obtain $\lambda_1 = \dfrac{7+\sqrt{57}}{2}$ and $\lambda_2 = \dfrac{7-\sqrt{57}}{2}$

(b) Find all the eigenvectors of the matrix A.

The eigenvectors $v_1 = (x_1, y_1)$ corresponding to λ_1 satisfy

$$\begin{bmatrix} 1-\lambda_1 & 4 \\ 2 & 6-\lambda_1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \lambda_1 \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

from which we get

$$\begin{cases} (1-\lambda_1)x_1 + 4y_1 = 0 \\ 2x_1 + (6-\lambda_1)y_1 = 0 \end{cases}$$

whose solution are $\{(t, \frac{1}{4}(\lambda_1 - 1)t) : t \in \mathbf{R}\}$.
The eigenvectors $v_2 = (x_2, y_2)$ corresponding to λ_2 can be obtained in a similar way.

(c) Find the null space of A.

The null space $\mathbf{N}$ of A is given by $N = \{x \in \mathbf{R}^2 : Ax = 0\}$. So we solve the system

$$\begin{cases} x + 4y & = 0 \\ 2x + 6y & = 0 \end{cases}$$

Thus the null space of A contains only the zero vector and equals $\{0\}$.

10 points 5. (a) Define what it means to say that the infinite series $\sum_{n=1}^{\infty} a_n$ converges.

We say the infinite series $\sum_{n=1}^{\infty} a_n$ converges if the sequence of partial sums

$$\sum_{k=1}^{n} a_n$$

converges to a real number.

(b) Determine if the series $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{1}{\sqrt{n}+\ln n}$ converges or not.

This is an alternating series and converges by alternating series test. That is if we set $a_n = \dfrac{1}{\sqrt{n}+\ln n}$ then

i. $a_n \geq 0$

ii. $a_n \geq a_{n+1}$

iii. $\displaystyle\lim_{n\to\infty} a_n = 0$

and all the conditions for the test are met.

(c) Determine if the series $\displaystyle\sum_{n=1}^{\infty}\frac{n!}{n^n}$ converges or not.

We will use the ratio test to show convergence.

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!n^n}{(n+1)^{n+1}n!} = \left(\frac{n}{n+1}\right)^n = \left(1+\frac{1}{n}\right)^{-n}$$

Thus

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \frac{1}{e} < 1$$

and thus the series converges.

10 points 6. (a) Define what it means to say a set of vectors is linearly dependent.

A set S of vectors is said to be linearly dependent if there exist scalars $c_1, c_2, \ldots, c_n$ not all zero and vectors $v_1, v_2, \ldots, v_n$ such that the linear combination $c_1 v_1 + c_2 v_2 + \ldots + c_n v_n = 0$.

(b) Let V be a set of vectors containing the zero vector. Is V linearly independent or dependent? Justify your answer.

A set of vectors S containing the zero vector $\mathbf{0}$ is a linearly dependent set. Indeed for any vectors $v_1, \ldots v_n$ and for any nonzero number c

$$c\mathbf{0} + c_1 v_1 + \ldots + c_n v_n = \mathbf{0}.$$

10 points 7. Let $I = \int_0^1 x \ln x \, dx$. Is I an improper integral? In either case evaluate I.

This is an improper integral since $\ln x$ is not defined at 0. So we have

$$\int_0^1 x \ln x \, dx = \lim_{t \to 0^+} \int_t^1 x \ln x \, dx$$

By using integration by parts

$$\int x \ln x dx = \frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 + C$$

Thus

$$\lim_{t \to 0^+} \int_t^1 x \ln x \, dx = -\frac{1}{4} - \lim_{t \to 0^+} [\frac{1}{2} t^2 \ln t - \frac{1}{4} t^2]$$

which after using L'Hopitals rule gives:

$$\lim_{t \to 0^+} \int_t^1 x \ln x \, dx = -\frac{1}{4}$$

10points 8. Set up a double or triple integral to represent the volume of the sphere $x^2 + y^2 + z^2 = 1$ and show the details to reach the answer $\frac{4\pi}{3}$.

The triple integral corresponding to the volume of sphere in spherical coordinates is:

$$\int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^2 \sin \phi d\rho d\phi d\theta = \frac{4\pi}{3}$$

10 points 9. For the sequence of functions $g_n(x) = \frac{1}{n} e^{-nx}$,

(a) Find the pointwise limit of the sequence.

Observe that for a fixed $x \geq 0$

$$\lim_{n\to\infty} \frac{1}{n}e^{-nx} = \lim_{n\to\infty} \frac{1}{ne^{nx}} = 0$$

Thus the function $g(x) = 0$ for $x \in [0, \infty]$ will be the point wise limit of the sequence. That is:

$$\lim_{n\to\infty} g_n(x) = 0$$

(b) Show that the sequence converges uniformly on $[0, \infty)$.

To show uniform convergence choose $\epsilon > 0$. Then

$$\left|\frac{1}{ne^{nx}}\right| < \frac{1}{n} < \epsilon$$

If we choose N so that $\frac{1}{N} < \epsilon$, then for all $n \geq N$ we have

$$\left|\frac{1}{ne^{nx}}\right| < \epsilon$$

for all $n \geq N$ and for all x.

10 points 10. (a) Give the definition of a Cauchy sequence.

A sequence $\{a_n\}$ is said to be a Cauchy sequence if for all $\epsilon > 0$ there exists a positive integer N such that $|a_n - a_m| < \epsilon$ for all $n, m > N$.

(b) Prove that every convergent sequence is a cauchy sequence.

Let $\{a_n\}$ be a sequence converging to a number L and $\epsilon > 0$. Then by definition of convergence there exists a positive integer N such that $|a_n - L| < \frac{\epsilon}{2}$ for all $n > N$. Thus for all $n, m > N$

$$|a_n - a_m| < |a_n + L - L - a_m| \leq |a_n - L| + |a_m - L| < \epsilon$$

completing the proof of the statement.

(c) Prove that every convergent sequence is bounded. Is the converse true? Prove or disprove.