# HOWARD UNIVERSITY
# DEPARTMENT OF MATHEMATICS
# SENIOR COMPREHENSIVE EXAMINATION
# OCTOBER 25, 2014

**Name:** ______________________________

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⇒ This exam consists of 10 questions. Answer all the questions in increasing numerical order in the provided bluebook. Each question is worth 10 points.

⇒ Show all your work as neatly and legibly as possible. Make your reasoning clear. In problems with multiple parts, be sure to go on to subsequent parts even if there is some part you cannot do.

| **Question** | **Points** | **Out of** |
|---|---|---|
| 1 | | 10 |
| 2 | | 10 |
| 3 | | 10 |
| 4 | | 10 |
| 5 | | 10 |
| 6 | | 10 |
| 7 | | 10 |
| 8 | | 10 |
| 9 | | 10 |
| 10 | | 10 |
| Total | | 100 |
| GRADE (P or F) | | |

10 points 1. Evaluate the following limits OR state if it does not exist. Justify your reasoning.

(a) $\lim_{x\to 0} \frac{5-5\cos x}{e^x - x - 1}$

This is an indeterminate form of 0/0 type. By using L'Hopitals rule repeatedly we get

$$\lim_{x\to 0} \frac{5\sin x}{e^x - 1} = \lim_{x\to 0} \frac{5\cos x}{e^x} = 1$$

(b) $\lim_{x\to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x}\right)$

First we have

$$\frac{x}{x-1} - \frac{1}{\ln x} = \frac{x\ln x - x + 1}{(\ln x)(x-1)}$$

So this limit is indeterminate form of type 0/0. So we use L'Hopitals rule and get:

$$\begin{aligned} \lim_{x\to 1} \frac{x\ln x - x + 1}{(\ln x)(x-1)} &= \lim_{x\to 1} \frac{\ln x}{(\frac{1}{x})(x-1) + \ln x} \\ &= \lim_{x\to 1} \frac{1/x}{(\frac{-1}{x^2})(x-1) + \frac{2}{x}} = \frac{1}{2} \end{aligned}$$

(c) $\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2}$

If we approach the origin along the curve $y = x^2$ we see that

$$\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2} = \lim_{(x,y)\to(0,0)} \frac{2x^4}{2x^4} = 1$$

However if we approach the origin along the curve $y = 2x^2$ we see that

$$\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2} = \lim_{(x,y)\to(0,0)} \frac{4x^4}{5x^4} = \frac{4}{5}$$

So this limit doesn't exist.

10 points 2. (a) Find all the points other than the origin at which the graph of $x^3 + y^3 - 9xy = 0$ has a horizontal tangent.

By using implicit differentiation we obtain:

$$\begin{aligned} 3x^2 + 3y^2y' - 9y - 9xy' &= 0 \\ y'(3y^2 - 9x) &= 9y - 3x^2 \end{aligned}$$

Therefore

$$y' = \frac{9y - 3x^2}{3y^2 - 9x}$$

We see that $y' = 0$ if $y = \frac{1}{3}x^2$

(b) Let $G(x) = \int_1^{e^{x^2}} \frac{1}{\sqrt{t}}\, dt$. Find the derivative of $G$ at $x = 2$, i.e., find $G'(2)$.
By using fundamental theorem of calculus and chain rule we get

$$G'(x) = \frac{2xe^{x^2}}{\sqrt{e^{x^2}}} = 2x\sqrt{e^{x^2}}$$

Thus $G'(2) = 4e^2$.

10 points 3. Evaluate the following integrals.

(a) $\int_0^{2\sqrt{\ln 3}} \int_{y/2}^{\sqrt{\ln 3}} e^{x^2}\, dxdy$
After changing order of integration we see that the integral is equal to

$$\int_0^{\sqrt{\ln 3}} \int_0^{2x} e^{x^2}\, dydx = \int_0^{\sqrt{\ln 3}} \int_0^{2x} e^{x^2}\, dydx = \int_0^{\sqrt{\ln 3}} 2xe^{x^2}\, dx$$

and this can easily be evaluated by substitution.

(b) $\int \frac{6x+7}{(x+2)^2}\, dx$
We use partial fractions method here:

$$\frac{6x+7}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}$$

From which we get that $A = 6$ and $B = -5$ and thus;

$$\int \frac{6x+7}{(x+2)^2}\, dx = 6\ln|x+2| + \frac{5}{x+2} + C$$

(c) $\int_C xydx + (x+y)\, dy$ along the curve $y = x^2$ from $(-1, 1)$ to $(2, 4)$
We can parametrize the curve as $x = t$ and $y = t^2$ for $t \in [-1, 2]$ and the line integral will be equal to:

$$\int_{-1}^{2} t^3\, dt + \int_{-1}^{2} 2t(t+t^2)\, dt =$$

10 points 4. Set up a double integral representing the surface area of a sphere and evaluate the integral.

Such a sphere is given by $x^2 + y^2 + z^2 = r^2$. The upper hemisphere is given by $z = \sqrt{a^2 - x^2 - y^2}$. So the double integral representing the surface area $A$ of a sphere of radius $r$ will be given by

$$2 \iint_D \sqrt{1 + f_x^2 + f_y^2}\, dA$$

where $f(x, y) = \sqrt{a^2 - x^2 - y^2}$ and $D$ is the projection of the upper hemisphere on to the $xy$ plane. So

$$f_x(x, y) = \frac{-x}{\sqrt{a^2 - x^2 - y^2}} \qquad \text{and} \qquad f_y(x, y) = \frac{-y}{\sqrt{a^2 - x^2 - y^2}}$$

From this

$$A = 2 \iint_D \sqrt{\frac{a^2}{r^2 - x^2 - y^2}}\, dA$$

and by using polar coordinates

$$A = 2 \int_0^{2\pi} \int_0^a \frac{a}{\sqrt{a^2 - r^2}}\, r\, dr\, d\theta$$

from which we obtain by substitution $A = 4\pi a^2$.

10 points 5. (a) Determine if the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 + 1}{2n^2 + n - 1}$ converges.

Note that

$$\lim_{n\to\infty} \frac{n^2 + 1}{2n^2 + n - 1} = \frac{1}{2} \neq 0$$

Thus, by divergence test this series diverges.

(b) Determine if the series $\displaystyle\sum_{n=1}^{\infty} \frac{(\sin n)(\ln n)^n}{n^n}$ converges.

We will check if the series converges absolutely:

$$\sum_{n=1}^{\infty} \left| \frac{(\sin n)(\ln n)^n}{n^n} \right| \leq \sum_{n=1}^{\infty} \left| \frac{(\ln n)^n}{n^n} \right|$$

and

$$\lim_{n\to\infty} \left| \frac{\ln n}{n} \right| = 0$$

So by root test the series converges absolutely and therefore converges.

(c) Determine the values of $x$ for which the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(3x-1)^n}{n^2}$ converges absolutely.

We note that

$$\left| \frac{(3x-1)^{n+1}}{(n+1)^2} \frac{n^2}{(3x-1)^n} \right| = \left( \frac{n}{n+1} \right)^2 |3x - 1|$$

and

$$\lim_{n to \infty} \left(\frac{n}{n+1}\right)^2 |3x-1| < 1$$

if $|3x-1| < 1$. Thus the series converges absolutely in the interval $(0, 2/3)$. At the end points $x = 0$ and $x = 2/3$ the series converges by comparing it to the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Thus the series converges for all $x$ in $[0, 2/3]$.

10 points

6. (a) Define what it means to say that: *$d$ is a metric on the set $X$.*

A metric $d$ on a set $X$ is a function $d : X \times X \to \mathbf{R}$ such that:

- $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$.
- $d(x, y) = d(y, x)$
- $d(x, y) \leq d(x, z) + d(z, y)$

for all $x$, $y$, and $z$ in $X$.

(b) For $\mathbf{x}, \mathbf{y} \in \mathbf{R}^2$, let $d(\mathbf{x}, \mathbf{y}) = \max(|x_1 - y_1|, |x_2 - y_2|)$, where $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$. Prove that $d$ is a metric on $\mathbf{R}^2$.

It is not difficult to verify the first two conditions in the definition of a metric given above. To prove the third condition let $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$ and $\mathbf{z} = (z_1, z_2)$ be in $\mathbf{R}^2$. Then by triangular inequality for absolute values:

$$|x_1 - y_1| \leq |x_1 - z_1| + |z_1 - y_1| \leq \max\{|x_1 - z_1|, |x_2 - z_2|\} + \max\{|z_1 - y_1|, |z_2 - y_2|\}$$

and

$$|x_2 - y_2| \leq |x_2 - z_2| + |z_2 - y_2| \leq \max\{|x_1 - z_1|, |x_2 - z_2|\} + \max\{|z_1 - y_1|, |z_2 - y_2|\}$$

Thus

$$\begin{aligned} d(\mathbf{x}, \mathbf{y}) = \max |x_1 - y_1|, |x_2 - y_2| &\leq \max\{|x_1 - z_1|, |x_2 - z_2|\} + \max\{|z_1 - y_1|, |z_2 - y_2|\} \\ &= d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z}) \end{aligned}$$

completing the proof.

10 points

7. Use mathematical induction to prove that $n! \leq n^n$ for any integer $n \geq 1$.

$1! \leq 1^1$.

Suppose $k! \leq k^k$.

Then

$$(k+1)! = (k+1)k! \leq (k+1)k^k \leq (k+1)(k+1)^k \leq (k+1)^{k+1}$$

and thus by principle of mathematical induction $k! \leq k^k$ for all $k$.

10 points 8. (a) Give the epsilon-delta definition of a what it means to say: *a function $f$ is continuous at the point $x = a$.*

$f$ is continuous at $x = a$ iff for every $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that $|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta$.

(b) By ONLY using the epsilon-delta definition, show that the function $f(x) = \dfrac{2x}{x+2}$ is continuous at $x = 2$.

Let $\epsilon > 0$.

$$\left|\frac{2x}{x+2} - f(2)\right| = \left|\frac{2x}{x+2} - 1\right| = \left|\frac{x-2}{x+2}\right| < \frac{1}{5}|x-2|$$

if $|x - 2| < 1$. Choose $\delta = \min\{1, 5\epsilon\}$.

10 points 9. (a) Define what it means to say *the vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}$ are linearly dependent.*

The vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}$ are said to be linearly dependent if whenever $c_1, \ldots, c_k$ are scalars with

$$c_1\mathbf{v_1} + c_2\mathbf{v_2} + \ldots + c_k\mathbf{v_k} = \mathbf{0}$$

we have $c_1 = c_2 = \ldots = c_k = 0$.

(b) Show that in the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 2 \\ 4 & 9 & 5 \end{bmatrix},$$

the row vectors are linearly dependent.

We need to show that the system

$$\begin{cases} x + y & = 0 \\ x + 3y + 2z & = 0 \\ 4x + 9y + 5z & = 0 \end{cases}$$

has a non zero solution. Solving the system of equations we obtain infinite solutions $(z, -z, z)$ where $z$ is any real number.

(c) Find the rank of the matrix $A$ in part (b).

We will reduce the matrix $A$ to row echelon form and the rank:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 2 \\ 4 & 9 & 5 \end{bmatrix} \underset{-4R1+R3\to R3}{\overset{R2-R1\to R1}{\longrightarrow}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 5 & 5 \end{bmatrix} \overset{R2/2\to R2}{\longrightarrow} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 5 & 5 \end{bmatrix} \overset{-5R2+R3\to R3}{\longrightarrow} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore $rank\ A = 2$.

10 points 10. (a) Let $U$ and $V$ be vector spaces over $\mathbf{R}$. Define what it means to say: *a mapping $F : U \to V$ is a linear transformation.*

$F : U \to V$ is a linear transformation if and only if:

- $F(\mathbf{u_1} + \mathbf{u_2}) = F(\mathbf{u}_1) + F(\mathbf{u}_2)$
- $F(c\mathbf{u}) = cF(\mathbf{u})$ for all $\mathbf{u_1},\ \mathbf{u_2}$ in $U$ and a scalar $c$.

(b) Suppose $F : \mathbf{R}^3 \to \mathbf{R}^2$ is defined by $F(x_1, x_2, x_3) = (x_1 + x_2 + x_3, 2x_1 - 3x_2 + 4x_3)$.

i. Find the matrix $A$ such that $F(\mathbf{x}) = A\mathbf{x}$ where $\mathbf{x} = (x_1, x_2, x_3)$.
We compute that $F(1, 0, 0) = (1, 2),\ F(0, 1, 0) = (1, -3)$, and $F(0, 0, 1) = (1, 4)$ therefore it is not difficult to see that $F(\mathbf{x}) = A\mathbf{x}$ if we take $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \end{bmatrix}$

ii. Show that $F$ is a linear transformation.
Matrix multiplication is distributive over matrix addition; therefore $F$ is a linear transformation.

iii. Describe the kernel of $F$.

iv. Find the nullity of $F$.