HOWARD UNIVERSITY
DEPARTMENT OF MATHEMATICS
SENIOR COMPREHENSIVE EXAMINATION
MARCH 22, 2014

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⇒ This exam consists of 10 questions. Answer all the questions in increasing numerical order in the provided bluebook. Each question is worth 10 points.

⇒ Show all your work as neatly and legibly as possible. Make your reasoning clear. In problems with multiple parts, be sure to go on to subsequent parts even if there is some part you cannot do.

Question	**Points**	**Out of**
1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total		100
GRADE (P or F)		

10 points 1. Evaluate the following limits OR state if it doesn't exist:

(a) $\lim_{x\to 5} \frac{2x^2-9x-5}{x^2-5x}$
This is a 0/0 type indeterminate form. One can use factoring or apply L'Hopitals rule. If we factor we get;

$$\lim_{x\to 5} \frac{(2x+1)(x-5)}{x(x-5)} = \lim_{x\to 5} \frac{2x+1}{x} = \frac{11}{5}$$

(b) $\lim_{x\to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x} \right)$
This is also an indeterminate form.
First we have

$$\frac{x}{x-1} - \frac{1}{\ln x} = \frac{x\ln x - x + 1}{(\ln x)(x-1)}$$

So this limit is indeterminate form of type 0/0. So we use L'Hopitals rule and get:

$$\begin{aligned} \lim_{x\to 1} \frac{x\ln x - x + 1}{(\ln x)(x-1)} &= \lim_{x\to 1} \frac{\ln x}{(\frac{1}{x})(x-1) + \ln x} \\ &= \lim_{x\to 1} \frac{1/x}{(\frac{-1}{x^2})(x-1) + \frac{2}{x}} = \frac{1}{2} \end{aligned}$$

(c) $\lim_{(x,y)\to(0,0)} \frac{x\sin(y)}{x^2+y^2}$ If we approach the origin along $y = 0$ we get $\lim_{(x,y)\to(0,0)} \frac{x\sin(y)}{x^2+y^2} = 0$

and if we approach origin along the line $y = x$ we get

$$\lim_{(x,y)\to(0,0)} \frac{x\sin(y)}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{y\sin(y)}{2y^2} = \frac{1}{2}$$

Therefore the limit doesn't exist.

10 points 2. (a) Give the definition of a function f that is differentiable at $x = a$.
A function f is said to be differentiable at $x = a$ if

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

exists.

(b) Let $f(x) = x|x|$. Either find $f''(0)$ if it exists, or prove that $f''(0)$ doesn't exist.

Note that we can rewrite the function as

$$f(x) = \begin{cases} x^2 & \text{if } \; x \geq 0 \\ -x^2 & \text{if } \; x < 0 \end{cases}$$

Thus it is easy to see that

$$f'(x) = \begin{cases} 2x & \text{if } \; x \geq 0 \\ -2x & \text{if } \; x < 0 \end{cases}$$

From this we can see that $f'(x) = 2|x|$ and by using the right and left derivatives at 0 it is not difficult to show that $f"(0)$ doesn't exist.

10 points 3. Evaluate the following integrals:

(a) $\displaystyle\int \frac{dx}{x^2+5x+6}$

By using partial fractions we obtain

$$\frac{1}{x^2+5x+6} = \frac{1}{x+2} - \frac{1}{x+3}$$

and by integrating we obtain

$$\int \frac{dx}{x^2+5x+6} = \int \frac{dx}{x+2} - \int \frac{dx}{x+3} = \ln\frac{|x+2|}{|x+3|} + C$$

(b) $\displaystyle\int_C xydx + (x-y)dy$ where C is the segment from $(1,0)$ to $(3,1)$.

We can parametrize the segment from $(1,0)$ to $(3,1)$ as $(t, \frac{1}{2}t - \frac{1}{2})$ for t in $[1,3]$. We then get

$$\int_C xydx + (x-y)dy = \int_1^3 \frac{t^2}{2} - \frac{t}{2}dt + \int_1^3 \frac{t}{2} + \frac{1}{2}dt = \frac{29}{6}$$

(c) $\displaystyle\int_0^8 \int_{y^{1/3}}^2 \frac{y^2e^{x^2}}{x^8}\,dx\,dy$ (Hint: Reverse the order of integration first)

If we reverse the order of operations we get:

$$\begin{aligned} \int_0^8 \int_{y^{1/3}}^2 \frac{y^2e^{x^2}}{x^8}\,dx\,dy &= \int_0^2 \int_0^{x^3} \frac{y^2e^{x^2}}{x^8}\,dy\,dx \\ &= \int_0^2 \frac{xe^{x^2}}{3}dx = \frac{1}{6}(e^2-1) \end{aligned}$$

10 points 4. Assume that $f(x)$ is a continuous function on $[0,1]$ and that $0 \leq f(x) \leq 1$. Prove that there exists a c in $(0,1)$ such that $f(c)=c$. State clearly any theorem(s) used in your proof.

Proof: Consider the function $g(x) = f(x) - x$ which is obviously continuous on $[0,1]$ since f is continuous. Then $g(0) = f(0) \geq 0$ and $g(1) = f(1) - 1 \leq 0$ since $0 \leq f(x) \leq 1$ for all x in $[0,1]$. By intermediate value theorem there exists a number c in $[0,1]$ such that $g(c) = 0$, which means that there is a number c in $[0,1]$ such that $f(c) = c$ and the proof is complete.

10 points 5. (a) Is $\int_0^1 x^2 \ln x \, dx$ an improper integral? Why or why not? Justify your answer.

An integral $\int_a^b f(x)dx$ is called improper if one or both of the following conditions is satisfied:

i the interval of integration is unbounded

ii. the function $f(x)$ has an infinite discontinuity at some point in $[a,b]$.

Based on this definition the integral is NOT improper. Note that $f(x) = x^2 \ln x$ is discontinuous at $x=0$ but this is not an infinite discontinuity since $\lim_{x \to 0^+} x^2 \ln x = 0$.

(b) Evaluate $\int_0^1 x^2 \ln x \, dx$.

$$\int_0^1 x^2 \ln x \, dx \qquad = \lim_{t \to 0^+} \int_t^1 x^2 \ln x \, dx$$

if we use integration by parts

$$= \lim_{t \to 0^+} \int_t^1 x^2 \ln x \, dx$$

$$= \lim_{t \to 0^+} \frac{x^3 \ln x}{3} - \frac{x^3}{9} \Big|_{x=t}^{x=1}$$

$$= -\frac{1}{9} - \lim_{t \to 0^+} \frac{t^3 \ln t}{3} - \frac{t^3}{9} = -\frac{1}{9}$$

by using L'Hopitals rule to find the limit.

10 points 6. For the matrix $A = \begin{bmatrix} -2 & 12 \\ -1 & 5 \end{bmatrix}$, do the following.

(a) Find all of the eigenvalues of the matrix A.

(b) Find all of the eigenvectors of the matrix A.

(c) Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

Solution: a) The eigenvalues are solutions of the equation

$$det(A - \lambda I) = det \begin{bmatrix} -2-\lambda & 12 \\ -1 & 5-\lambda \end{bmatrix} = 0$$

This gives the equation:

$$(-2-\lambda)(5-\lambda) + 12 = 0$$

Expanding and solving we obtain two distinct values for λ given as $\lambda_1 = 2$ and $\lambda_2 = 1$

b) The eigenvectors $v_1 = (x_1, y_1)$ corresponding to $\lambda_1 = 2$ satisfy

$$\begin{bmatrix} -2 & 12 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = 2 \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

from which we get

$$\begin{cases} 4x_1 - 12y_1 = 0 \\ x_1 - 3y_1 = 0 \end{cases}$$

whose solution are $\{(t, \frac{1}{3}t) : t \in \mathbf{R}\}$. The eigenvectors are nonzero vectors of the form $(t, \frac{1}{3}t)$.

The eigenvectors $v_2 = (x_2, y_2)$ corresponding to $\lambda_2 = 1$ are nonzero vectors of the form $(4t, t)$. c) Since A has two distinct eigenvalues it is diagonalizable. A basis for the eigenvalue $\lambda_1 = 2$ could be $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and a basis for the eigenvalue $\lambda_2 = 1$ could be $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

So the matrix $P = \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}$ diagonalizes the matrix A. That is

$$A = P \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} P^{-1}.$$

10 points

7. (a) Let $T : \mathbf{R}^2 \to \mathbf{R}^3$ be a linear transformation with

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -2 \\ -7 \end{bmatrix}.$$

Find a matrix representation of T. That is, find the matrix A such that $T\mathbf{x} = \mathbf{Ax}$ for all x in $\mathbf{R}^2$.

(b) Let $T : V \to W$ be a linear transformation from vector space V into vector space W. Prove that the null space of T is a subspace of V.

Solutions;

a) Since $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are the basis of $\mathbf{R}^2$ the matrix representation of T is simply the 3×2 matrix given by

$$\begin{bmatrix} -1 & 1 \\ 3 & -2 \\ 5 & -7 \end{bmatrix}$$

b) The null space of T is defined as $N = \{x \in \mathbf{R}^2 : T\mathbf{x} = \mathbf{0}\}$. To show that N is a subspace, it suffices to show that

i. $N \neq \emptyset$

ii. for $\mathbf{x_1}$ and $\mathbf{x_2}$ in N $\mathbf{x_1} - \mathbf{x_2}$ is in N.

iii. for $\mathbf{x}$ in N and a real number c, $c\mathbf{x}$ is N.

(i) holds trues since $T\mathbf{0} = \mathbf{0}$
(ii) By linearity of T we know that $T(\mathbf{x_1} - \mathbf{x_2}) = \mathbf{T}(\mathbf{x_1}) - \mathbf{T}(\mathbf{x_2}) = \mathbf{0}$. (iii) Again by linearity of T we have $T(c\mathbf{x_1}) = cT(\mathbf{x_1}) = \mathbf{0}$.

10 points 8. (a) Prove that an absolutely convergent series is convergent.

(b) Determine if the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(e^{\pi n^2})}{\sqrt{n^5 + n^2}}$ converges or diverges.

(c) Determine if the series $\displaystyle\sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$ converges or diverges.

Solutions:

a) Suppose $\displaystyle\sum_{n=1}^{\infty} a_n$ is absolutely convergent. Then $\displaystyle\sum_{n=1}^{\infty} |a_n|$ converges. To prove that $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, it suffices to show that the sequence of partial sums is Cauchy sequence. Indeed for any $\epsilon > 0$ there exists N such that

$$|\sum_{n=1}^{m} a_n - \sum_{n=1}^{l} a_n| \leq \sum_{n=m}^{l} |a_n| < \epsilon$$

for $l \geq m \geq N$. This completes the proof.

b) Note that since $-1 \leq \sin x \leq 1$ for any x,

$$\sum_{n=1}^{\infty}\left|\frac{\sin(e^{\pi n^2})}{\sqrt{n^5+n^2}}\right| \le \sum_{n=1}^{\infty}\frac{1}{\sqrt{n^5+n^2}} \le \sum_{n=1}^{\infty}\frac{1}{\sqrt{n^5}}$$

and the later series is convergent as it is a p series with $p = 5/2 > 1$. This proves the series is absolutely convergence and hence is convergent by part(a).

c) $\lim_{n\to\infty} \cos\frac{1}{n} = 1$ and by divergence test the series is divergent.

10 points 9. For the sequence of functions $g_n(x) = \frac{x}{n}e^{-x/n}$,

(a) Find the pointwise limit of the sequence.

(b) Show that the sequence converges uniformly on $[0, 500]$.

Solutions:

a) Fix x. Then $\lim_{n\to\infty} \frac{x}{n}e^{-x/n} = \lim_{n\to\infty} \frac{x}{n\, e^{x/n}} = 0$. Thus for each x we see that $\lim_{n\to\infty} g_n(x) = 0$.

b) We claim that g_n converges uniformly to $g(x) = 0$ on $[0, 500]$.

To show the uniform convergence choose $\epsilon > 0$. Then

$$|g_n(x) - g(x)| = \left|\frac{x}{ne^{x/n}}\right| < \frac{x}{n} < \frac{500}{n}$$

for x in $[0, 500]$. If we choose N so that $\frac{500}{N} < \epsilon$, then for all $n \ge N$ we have

$$\left|\frac{x}{ne^{x/n}}\right| < \epsilon$$

for all $n \ge N$ and for all x.

10 points 10. Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function. For each of the following, either prove that the statement is true or provide a counterexample.

(a) If $A \subseteq \mathbf{R}$ is closed, then $f(A)$ is closed.

(b) If x_n is a sequence in $\mathbf{R}$ and $x_n \to x$, then $f(x_n) \to f(x)$.

(c) If $B \subseteq \mathbf{R}$ is open, then $f^{-1}(B)$ is open in $\mathbf{R}$.

Solutions:

a) False. Take $f(x) = e^x$ with $A = (-\infty, \infty)$. A is a closed set in $\mathbf{R}$ and $f(A) = (0, \infty)$ which is NOT a closed set.

b) True.

Let $\epsilon > 0$. We need to show that there exist N such that $|f(x_n) - f(x)| < \epsilon$ whenever $n > N$.

Since f is continuous at x there exists a $\delta > 0$ such that $|f(x_n) - f(x)| < \epsilon$ whenever

$|x_n - x| < \delta$. Also, since $x_n \to x$ there exists an N_o such that $|x_n - x| < \delta$ whenever $n > N_o$.
Choose $N = N_o$. Then for $n > N$ we have $|x_n - x| < \delta$ and consequently $|f(x_n) - f(x)| < \epsilon$.
c) True.
Recall that a subset O of $\mathbf{R}$ is open if for each $z \in O$ there is an open interval I such that $z \in I \subseteq O$.

To show that $f^{-1}(B)$ is open, let $x_o \in f^{-1}(B)$ and $f(x_o) = y_o$. Choose $\epsilon > 0$ such that the interval $(y_o - \epsilon, y_o + \epsilon) \subseteq B$. This possible since B is an open set. Then by continuity of f at x_o there exists a $\delta > 0$ such that if $|x - x_o| < \delta$ then $|f(x) - y_o| < \epsilon$.
Set $I = (x_o - \delta, x_o + \delta)$. Then $x_o \in I$ and $f(I) \subseteq (y_o - \epsilon, y_o + \epsilon) \subseteq B$ and hence we have

$$x_o \in I \subseteq f^{-1}((y_o - \epsilon, y_o + \epsilon)) \subseteq f^{-1}(B)$$

and the proof is complete.