HOWARD UNIVERSITY
DEPARTMENT OF MATHEMATICS
SENIOR COMPREHENSIVE EXAMINATION
MARCH 22, 2014

Name: ____

Id. Number: __________________________________

Email address: _______________________________

Address: ____

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Signature: ____________________________________

⇒ This exam consists of 10 questions. Answer all the questions in increasing numerical order in the provided bluebook. Each question is worth 10 points.

⇒ Show all your work as neatly and legibly as possible. Make your reasoning clear. In problems with multiple parts, be sure to go on to subsequent parts even if there is some part you cannot do.

Question	**Points**	**Out of**
1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total		100
GRADE (P or F)		

$\boxed{\text{10 points}}$ 1. Evaluate the following limits OR state if it doesn't exist:

(a) $\displaystyle\lim_{x\to 5} \frac{2x^2-9x-5}{x^2-5x}$

(b) $\displaystyle\lim_{x\to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x}\right)$

(c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x\sin(y)}{x^2+y^2}$

$\boxed{\text{10 points}}$ 2. (a) Give the definition of a function f that is differentiable at $x = a$.

(b) Let $f(x) = x|x|$. Either find $f''(0)$ if it exists, or prove that $f''(0)$ doesn't exist.

$\boxed{\text{10 points}}$ 3. Evaluate the following integrals:

(a) $\displaystyle\int \frac{dx}{x^2+5x+6}$

(b) $\displaystyle\int_C xy\,dx + (x-y)\,dy$ where C is the segment from $(1,0)$ to $(3,1)$.

(c) $\displaystyle\int_0^8 \int_{y^{1/3}}^2 \frac{y^2 e^{x^2}}{x^8}\,dx\,dy$ (Hint: Reverse the order of integration first)

$\boxed{\text{10 points}}$ 4. Assume that $f(x)$ is a continuous function on $[0,1]$ and that $0 \le f(x) \le 1$. Prove that there exists a c in $(0,1)$ such that $f(c) = c$. State clearly any theorem(s) used in your proof.

$\boxed{\text{10 points}}$ 5. (a) Is $\displaystyle\int_0^1 x^2 \ln x\,dx$ an improper integral? Why or why not? Justify your answer.

(b) Evaluate $\displaystyle\int_0^1 x^2 \ln x\,dx$.

$\boxed{\text{10 points}}$ 6. For the matrix $A = \begin{bmatrix} -2 & 12 \\ -1 & 5 \end{bmatrix}$, do the following.

(a) Find all of the eigenvalues of the matrix A.

(b) Find all of the eigenvectors of the matrix A.

(c) Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

$\boxed{\text{10 points}}$ 7. (a) Let $T : \mathbf{R}^2 \to \mathbf{R}^3$ be a linear transformation with

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -2 \\ -7 \end{bmatrix}.$$

Find a matrix representation of T. That is, find the matrix A such that $T\mathbf{x} = \mathbf{A}\mathbf{x}$ for all x in $\mathbf{R}^3$.

(b) Let $T : V \to W$ be a linear transformation from vector space V into vector space W. Prove that the null space of T is a subspace of V.

$\boxed{\text{10 points}}$ 8. (a) Prove that an absolutely convergent series is convergent.

(b) Determine if the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(e^{\pi n^2})}{\sqrt{n^5 + n^2}}$ converges or diverges.

(c) Determine if the series $\displaystyle\sum_{n=1}^{\infty} \cos\left(\frac{1}{n}\right)$ converges or diverges.

$\boxed{\text{10 points}}$ 9. For the sequence of functions $g_n(x) = \dfrac{x}{n} e^{-x/n}$,

(a) Find the pointwise limit of the sequence.

(b) Show that the sequence converges uniformly on $[0, 500]$.

$\boxed{\text{10 points}}$ 10. Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function. For each of the following, either prove that the statement is true or provide a counterexample.

(a) If $A \subseteq \mathbf{R}$ is closed, then $f(A)$ is closed.

(b) If x_n is a sequence in $\mathbf{R}$ and $x_n \to x$, then $f(x_n) \to f(x)$.

(c) If $B \subseteq \mathbf{R}$, then $f^{-1}(B)$ is open in $\mathbf{R}$.