Calculus I Final Exam
April 27, 2021
Howard University Mathematics Department

MUST GIVE STEP BY STEP EXPLANATIONS TO GET CREDIT FOR ANSWERS.
No calculators or electronic devices are permitted.

PART I

Do ALL 3 problems. EACH WORTH 16 POINTS.

1. Consider the function $f(x) = \cos(x) + \sin(x)$.

   (a) Determine the critical number(s) of $f(x)$ on the interval $[0, 2\pi]$. Note: $\tan x = 1$ only at $\frac{\pi}{4}, \frac{5\pi}{4}$ in $[0, 2\pi]$.

   (b) What is the global maximum of $f(x)$ on the interval $[0, 2\pi]$? What is the global minimum of $f(x)$ on this same interval?
   You can use $\sin(\pi/4) = \cos(\pi/4) = \sqrt{2}/2, \sin(5\pi/4) = \cos(5\pi/4) = -\sqrt{2}/2$.

2. Compute the derivative of the following functions using appropriate derivative rules:

   (a) $f(x) = x^3 \ln(x)$.

   (b) $g(x) = \cos(x^3 - 1) + 2^{3x}$.

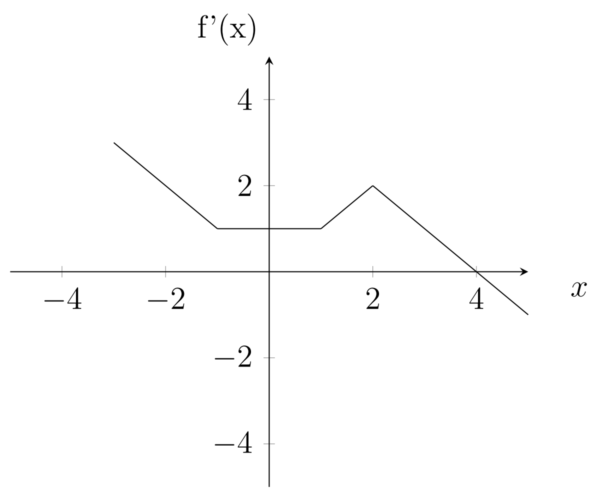
3. The function $f(x)$ has domain $[-3, 5]$. The graph below represents $f'(x)$.

   (a) Where is $f(x)$ increasing and where is it decreasing?

   (b) Where is $f(x)$ concave up and where is it concave down?
   [HINT: When graph is concave up what happens to $f'(x)$? What happens when it is concave down ?]

   (c) Sketch a possible graph of $f(x)$.

NOTE: THIS IS GRAPH OF DERIVATIVE $f'(x)$, NOT $f(x)$



PART II

Choose any 8 problems. EACH WORTH 16 POINTS.

1. (a) Find the domain of $f(x) = \sqrt{4x^2 - x} + 2x$ and express your answer in interval notation.

   (b) Find the inverse function $f^{-1}(x)$ of $f(x) = \ln(x-1)$ and check that $f^{-1} \circ f(x) = x$.

2. Given

$$f(x) = \begin{cases} \dfrac{\sqrt{2x+3}-3}{x-3} & if\ x > 3 \\ \\ \dfrac{10}{x^2+21} & if\ x \leq 3 \end{cases}$$

   Show that $f(x)$ is continuous on the interval $(-\infty, \infty)$.

3. The population of a town after t years is given by $P(t) = \dfrac{1200}{1+e^{-0.1t}}$. Find the rate of change of this population after t years. Is it always increasing or decreasing? What happens to the population as t approaches ∞ ?

4. Show that $2x - \sin x$ can have at the most one real number as zero (i.e, x-intercept) in any interval $[a, b]$ using Rolle's theorem. State the conditions for Rolle's theorem and how they are satisfied here.

5. Differentiate $y = \dfrac{(x-2)^3}{\sqrt{x^2+1}}$ and $y = x^{x-1}$ using logarithmic differentiation.

6. Use implicit differentiation to find the equation of the tangent line to the ellipse $\dfrac{x^2}{2} + \dfrac{y^2}{8} = 1$ at $(1, 2)$.

7. Find the linearization (linear approximation) $L(x)$ of $f(x) = e^x$ at $a = 0$ and use it to approximate $e^{0.1}$.

8. Evaluate the following limits, or show that they do not exist.

   (a) $\lim\limits_{x \to 0} \dfrac{x^3 + x^2}{\cos(2x) - 1}$

   (b) $\lim\limits_{x \to \infty} (x + x^2)^{1/x}$

9. You are building a toolbox that is to have a square base and an open top. The total volume of the box shall be 2 cubic meters. The materials used for the base cost \$3 per square meter, and the materials for the sides cost \$6 per square meter. Determine the minimum cost of production for the toolbox.

10. Solve for y using integration by substitution: $y = \displaystyle\int e^{\cos x} \sin x \; dx$.

11. Evaluate the following integrals (Calculate as much as you can):

$$(a) \int_0^1 \sqrt{t} - t^2 \, dt \qquad (b) \int_1^2 \frac{e^2}{x^3} \, dx$$

12. Find the Riemann sum approximation for $\int_0^3 \frac{1}{1+x^2}, dx$ using 3 intervals and left endpoints. Then do the same using 3 intervals and right endpoints. Find the actual value of this integral, given that $tan^{-1}(3) = 1.25$. How does the approximation compare to the actual value? What about the average of the two Riemann sum approximations?