Final Examination
Calculus I (MATH-156), Fall 2005

Max Points: 200

1. Draw the graph of the function with domain $[0, 4]$ and having the following properties:

   (i) $f(0) = -1$, $f(2) = 2$, and $f(4) = 1$.

   (ii) $\lim_{x \to 1^-} f(x) = 1$.

   (iii) $\lim_{x \to 1^+} f(x) = 3$.

   (iv) $\lim_{x \to 4^-} f(x) = 0$.

2. (a) Prove that $\lim_{x \to -9} \left(\frac{x^2 - 81}{x + 9} \right) = -18$.

   (b) Suppose $\lim_{x \to \sqrt{2}} f(x) = 3$, and $\lim_{x \to \sqrt{2}} (fg)(x) = -\sqrt{2}$. show that $\lim_{x \to \sqrt{2}} g(x)$ exists and find $\lim_{x \to \sqrt{2}} g(x)$.

3. Show that
$$f(x) = \begin{cases} x^2 + 3, & x \leq 1 \\ x + 3, & x > 1 \end{cases}$$

   is continuous , but not differentiable at $x = 1$. Sketch the graph of f.

4. Find the indicated derivative. Simplify your answers.

   (i) $f(x) = \frac{(3 - 2x)^{4/3}}{x^2}$, find $f'(x)$.

   (ii) $f(x) = \ln \frac{x^3}{\cos x}$, find $f'(x)$.

   (iii) $f(x) = \pi^{(\sin x - \cos x)}$, find $f'(\frac{\pi}{4})$.

   (iv) $f(x) = (\ln x) e^{3x}$, find $f'(1)$.

5. Evaluate the limit

   (i) $\lim_{x \to 0} \frac{x^2 e^x}{\sin^2 3x}$.

   (ii) $\lim_{x \to \infty} x \sin(\frac{\pi}{x})$.

6. (a) Use implicit differentiation to find $\frac{dy}{dx}$:
$$x^3 + y^3 = 3xy^2$$

   (b) Assume that x and y are differentiable functions of t. Find $\frac{dy}{dt}$ in terms of x, y and $\frac{dx}{dt}$:
$$x \sin y = 2$$

7. (a) Find the absolute extrema values of
$$f(x) = (x - 2)^2 (x + 3)^2 \quad \text{on} \quad (-\infty, \infty).$$

   Compute also inflection points (if any), and provide a brief sketch of the curve.

   (b) Given the polynomial $f(x) = x^6 - 2x^2 + x$ on the interval $[0, 2]$, show that the polynomial $6x^5 - 4x + 1$ has a zero on the open interval $(0, 2)$.

8. A container with square base, vertical sides and open top is to be made from 1000 ft^2 of the material. Find the dimensions of the container with greatest volume.

9. Find

   (a) $\int 3x^2 (x^3 + 5)^9 \, dx$

   (b) $\int \sin^4 x \cos x \, dx$

10. (a) Evaluate $\int_1^3 f(x) \, dx$, when
$$f(x) = \begin{cases} x & \text{for} \quad 1 \leq x \leq 2 \\ 2 & \text{for} \quad 2 \leq x \leq 3 \end{cases}$$

   (b) Let
$$g(x) = \int_0^x \frac{\cos t}{t^2 + 3} \, dt$$

   Find (i) $g'(0)$ and (ii) $g''(0)$.