Math 180 Linear Algebra Final

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1. Describe the solution set of the given homogeneous system in parametric vector form **x**=s**u**+t**v**: (10 pts)

$$\begin{aligned} x_1 + x_2 - 2x_3 + 4x_4 &= 0 \\ 2x_1 + x_2 - 4x_3 + 5x &= 0 \\ 3x_1 + 2x_2 - 6x_3 + 9x_4 &= 0 \end{aligned}$$

2. Let $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{y}_1 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \mathbf{y}_2 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$. If T is a linear transformation that maps $\mathbf{e}_1$ to $\mathbf{y}_1$, and $\mathbf{e}_2$ to $\mathbf{y}_2$, find the image of $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. (10 pts)

3.a. Find the inverse of

$$\begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}.$$

(10 pts)

b. Use the inverse found in part a to solve the system (10 pts)

$$\begin{aligned} x_1 + 4x_2 &= 17 \\ 2x_1 - x_2 &= 7 \end{aligned}$$

4 Which of the following are subspaces of the given vector space? Justify your answer.

a. Vectors in $\mathbf{R}^3$ of the form $(a, b, a+2b)$ (10 pts)

b. Vectors in $\mathbf{R}^3$ of the form $(a, b, a+4)$ (10 pts)

c. 2×2 matrices whose (1,2) entry is zero (i.e., the top, right-hand corner entry) (10 pts)

5. Find the determinant. (10 pts)

$$\begin{vmatrix} 5 & -2 & 4 \\ 0 & 3 & -5 \\ 2 & -4 & 1 \end{vmatrix}$$

6. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation determined by the matrix $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$, where *a*, *b*, and *c* are positive numbers. Let *S* be the unit ball, whose bounding surface has equation $x_1^{\,2} + x_2^{\,2} + x_3^{\,2} = 1$.

a. Show that *T(S)* is bounded by the ellipsoid with the equation $\dfrac{x_1^{\,2}}{a^2} + \dfrac{x_2^{\,2}}{b^2} + \dfrac{x_3^{\,2}}{c^2} = 1$. (10 pts)

b. Use the fact that the volume of the unit ball is $4\pi/3$ to determine the volume of the region bounded by the ellipsoid in part (a). (10 pts)

7.a. Let W be the subspace spanned by the **u**'s, and write **y** as the sum of a vector in W and a vector orthogonal to W. (10 pts)

$$\mathbf{y} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \ \mathbf{u}_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \ \mathbf{u}_2 = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}$$

b. Find a unit vector in the same direction as $\mathbf{u}_1$. (10 pts)

8. Let A be a matrix such that $AA^T = \mathbf{0}$ (the zero matrix). Show that $A = \mathbf{0}$. (Hint: Consider the diagonal elements.) (10 pts)

9. Given that the eigenvalues of the symmetric matrix A are 5, 2, and -2, find an orthogonal matrix P and a diagonal matrix D that orthogonally diagonalize A. (10 pts)

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & 1 \\ 3 & 1 & 1 \end{bmatrix}.$$

10.a. Classify the quadratic form $3x_1^2 - 4x_1x_2 + 6x_2^2$ as positive definite, negative definite or indefinite. (10 pts)

b. Make a change of variable $\mathbf{x} = P\mathbf{y}$ that transforms the quadratic form into one with no cross-product term. Write the new quadratic form. (10 pts)

12. Find the characteristic polynomial and the eigenvalues of $\begin{bmatrix} 3 & -2 \\ -3 & 4 \end{bmatrix}$. (10 pts)

13. Which of the following matrices are orthogonal? For those that are orthogonal, find the inverses. (20 pts)

a. $\begin{bmatrix} .6 & .8 \\ .8 & -.6 \end{bmatrix}$

b. $\begin{bmatrix} -5 & 2 \\ 2 & 5 \end{bmatrix}$

c. $\begin{bmatrix} .5 & .5 & -.5 & -.5 \\ -.5 & .5 & -.5 & .5 \\ .5 & .5 & .5 & .5 \\ -.5 & .5 & .5 & -.5 \end{bmatrix}$

14. Find the maximum of the following quadratic form over the set of unit vectors:

$Q(\mathbf{x}) = 8x_1^{\ 2} + 7x_2^{\ 2} - 3x_3^{\ 2}$ (10 pts)

EXTRA CREDIT

15. Suppose that a university must remodel x classrooms and y dormitory rooms over the summer. It s more cost-effective to work simultaneously on both projects than only one. Because the university's resources are limited, x and y must satisfy

$$25x^2 + 4y^2 \leq 100$$

The university would like to make the changes in a way that would best please the student population. Therefore, it would like to maximize the indifference curve q(x,y) = xy under this constraint. What values of x and y would do this? (Hint: substitute $\dfrac{x}{2} = x_1$ and $\dfrac{y}{5} = y_1$ into the constraint inequality.)