Calculus III   Final Exam Form A Solution Key
Fall 2025

1. A particle traveling in a circle has velocity function $v(t) = 3\sin t\mathbf{i} + 3\cos t\mathbf{j}$ with initial position $\mathbf{r}(0) = 3\mathbf{j} - 3\mathbf{k}$.

   (a) Find the position function $\mathbf{r}(t)$ for this particle.

   (b) Find the distance traveled by this particle over $0 \le t \le \frac{\pi}{2}$.

(a) $\mathbf{r}'(t) = 3\sin t\mathbf{i} + 3\cos t\mathbf{j}$, $\mathbf{r}(0) = 3\mathbf{j} - 3\mathbf{k}$.

$$\begin{aligned} \int \mathbf{r}'(t)dt &= \int (3\sin t\mathbf{i} + 3\cos t\mathbf{j})dt \\ \mathbf{r}(t) &= -3\cos t\mathbf{i} + 3\sin t\mathbf{j} + \mathbf{c} \\ \mathbf{r}(0) &= -3\cos(0)\mathbf{i} + 3\sin(0)\mathbf{j} + \mathbf{c} \\ 3\mathbf{j} - 3\mathbf{k} &= -3\mathbf{i} + \mathbf{c} \\ 3\mathbf{i} + 3\mathbf{j} - 3\mathbf{k} &= \mathbf{c}. \end{aligned}$$

Therefore, $\mathbf{r}(t) = (-3\cos t + 3)\mathbf{i} + (3\sin t + 3)\mathbf{j} - 3\mathbf{k}$.

(b)

$$\begin{aligned} TD &= \int_0^{\frac{\pi}{2}} \big|\mathbf{v}(t)\big|dt = \int_0^{\frac{\pi}{2}} \sqrt{(3\sin t)^2 + (3\cos t)^2}\ dt \\ &= \int_0^{\frac{\pi}{2}} \sqrt{9(\sin^2 t + \cos^2 t)}dt = \int_0^{\frac{\pi}{2}} 3\ dt = \big[3t\big]_0^{\frac{\pi}{2}} = \tfrac{3\pi}{2}. \end{aligned}$$

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2. Do the following, where $z = f(x, y)$ where $x = r\cos 2\theta$, $y = r\sin 2\theta$.

   (a) Using limit notation, write what it means for the function f_y to be continuous at the point (a, b).

   (b) Find $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$ if f_x and f_y are continuous.

   (c) Show that $y = r\sin 2\theta$ is a solution to the partial differential equation $4(y_r)^2 + (y_{\theta r})^2 + 4(y_{r\theta})^2 + (y_{r\theta\theta})^2 = 20$.

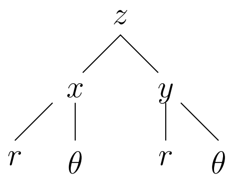
(a) The function f_y is continuous at the point (a, b) means that:

(i) $f_y(a, b)$ exists

(ii) $\lim\limits_{(x,y)\to(a,b)} f_y(x, y)$ exists

(iii) $\lim\limits_{(x,y)\to(a,b)} f_y(x, y) = f_y(a, b)$.

(b)



$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial r}$$
$$= f_x(x,y)\cos 2\theta \;+\; f_y(x,y)\sin 2\theta.$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial \theta}$$
$$= -2 f_x(x,y) r \sin 2\theta \;+2\; f_y(x,y) r \cos 2\theta.$$

(c)
$y_r = \sin 2\theta$, $y_{r\theta} = 2\cos 2\theta$, $y_{\theta r} = 2\cos 2\theta$, $y_{r\theta\theta} = -4\sin 2\theta$

$$\begin{aligned} 4(y_r)^2 + (y_{\theta r})^2 + 4(y_{r\theta})^2 + (y_{r\theta\theta})^2 &= 4\sin^2 2\theta + 4\cdot 4cos^2 2\theta + 4\cos^2 2\theta + 16\sin^2 2\theta \\ &= 4(\sin^2 2\theta + \cos^2 2\theta) + 16(\cos^2 2\theta + \sin^2 2\theta) \\ &= 4\cdot 1 + 16\cdot 1 = 20. \end{aligned}$$

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3. Let $f(x,y) = xy + e^{x+3y}$ denote the temperature of a circular disk in °F. Assume distance is measured in feet.

   (a) Find the equation of the tangent plane to $z = f(x,y)$ at $(-3,1)$.

   (b) Find $D_u f(-3,1)$ in the direction of $(1,4)$, indicate units of measure, and explain what $D_u f(-3,1)$ means.

(a)
$f_x = y + e^{x+3y}$, $f_y = x + 3e^{x+3y}$; so therefore
$f(-3,1) = -2$, $f_x(-3,1) = 2$, $f_y(-3,1) = 0$. The equation of the tangent plane is therefore

$$\begin{aligned} z &= f(-3,1) + f_x(-3,1)(x+3) + f_y(-3,1)(y-1) \\ &= -2 + 2(x+3) + 0(y-1) \\ z &= 2x + 4. \end{aligned}$$

(b)
$\mathbf{v} = \langle 1,4\rangle - \langle -3,1\rangle = \langle 4,3\rangle$. $|\mathbf{v}| = \sqrt{4^2+3^2} = 5$ and $\nabla f(-3,1) = \langle 2,0\rangle$.
Therefore $\mathbf{u} = \langle \frac{4}{5}, \frac{3}{5}\rangle$ and so

$$D_u f(-3,1) = \nabla f(-3,1)\cdot \mathbf{u} = \langle 2,0\rangle \cdot \langle \tfrac{4}{5}, \tfrac{3}{5}\rangle = \tfrac{8}{5}\,\tfrac{°\text{F}}{\text{ft}}. \tag{1}$$

The answer in (4) means that the temperature of the circular disk will increase by 8°F over a 5ft increase in the direction of $\mathbf{v}$ from the point $(-3,1)$.

4. The plane $z = 2x + 8y$ intersects the cylinder $x^2 + y^2 = 17$ in an ellipse. Find the maximum and minimum perpendicular distances from this ellipse to the xy-plane.

The goal is to optimize $f(x, y) = 2x + 8y$ subject to the constraint $g(x, y) = 17$ where $g(x, y) = x^2 + y^2$. By the method of Lagrange multipliers, we have that

$$\begin{aligned} \nabla f(x, y) &= \lambda \nabla g(x, y) \\ f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} &= \lambda g_x(x, y)\mathbf{i} + \lambda g_y(x, y)\mathbf{j}, \end{aligned}$$

which implies that

$$\begin{aligned} f_x(x, y) &= \lambda g_x(x, y) & \qquad f_y(x, y) &= \lambda g_y(x, y) \\ 2 &= \lambda \cdot 2x & 8 &= \lambda \cdot 2y \\ \frac{1}{\lambda} &= x & \frac{4}{\lambda} &= y \end{aligned} \tag{2}$$

By substitution into $g(x, y) = 17$ we obtain that

$$\begin{aligned} x^2 + y^2 &= 17 \\ \frac{1}{\lambda^2} + \frac{16}{\lambda^2} &= 17 \\ \lambda^2 &= 1 \end{aligned}$$

By substitution into (5) the two possibilities $\lambda = 1$, $\lambda = -1$ respectively correspond to the points $(-1, -4)$ and $(1, 4)$ on the circle $x^2 + y^2 = 17$. Thus the maximum and minimum values of $f(x, y)$ subject to $g(x, y) = 17$ are $f(-1, -4) = -34$ and $f(1, 4) = 34$. This means that the points on the ellipse that are the greatest distance from the xy-plane are $(-1, -4, -34)$ and $(1, 4, 34)$, and so 34 is the maximum distance from the ellipse to the xy-plane.

Since $f(x, y)$ is continuous over the domain $\{(x, y) \mid g(x, y) = 17\}$ and $f(-1, -4) = -34 < 0 < 34 = f(1, 4)$ the Intermediate Value Theorem implies that there is at least one point on the ellipse where $f(x, y) = 0$; and hence the minimum distance from the ellipse to the xy-plane is zero. To find where this occurs, set

$$\begin{aligned} f(x, y) = 2x + 8y &= 0 \text{ , which implies that} \\ x &= -4y \text{ , and so by substitution into } g(x, y) = 17 \\ (-4y)^2 + y^2 &= 17 \ \Rightarrow \\ y &= \pm 1 \text{ , } x = \mp 4 \end{aligned}$$

Therefore, the minimum distance of zero from the ellipse to the xy-plane occurs at the two points $(-4, 1, 0)$ and $(4, -1, 0)$.

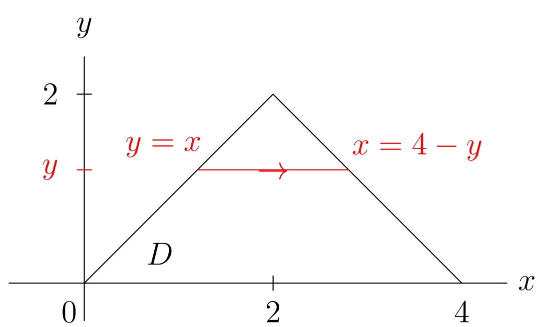
5. Let D be the triangular region in the xy-plane with vertices at $(0,0,0), (2,2,0)$, and $(4,0,0)$, which is bounded by $y = x$, $x + y = 4$ and $y = 0$. Do the following.

   (a) Show that $\iint\limits_D y\ dA = \frac{8}{3}$ by evaluating this integral.

   (b) Describe a solid which the integral in (a) represents the volume of.

   (c) If C is the boundary of D oriented counter-clockwise, use Green's Theorem and your answer in part (a) to evaluate the line integral $\int\limits_C x^2\ dx\ +\ xy\ dy$.

   (d) Let $-C$ be the traversal of C in a clockwise direction. Use your answer in (c) to set up and state the value of a line integral over $-C$ that equals the work done by $\mathbf{F} = x^2\mathbf{i} + xy\mathbf{j}$ in moving a particle along $-C$ (in a clock-wise direction).



(a)
As a type II region, we can express the domain of integration as $D = \{(x, y) \mid 0 \leq y \leq 2\ ,\ y \leq x \leq 4 - y\}$. Using the limits prescribed by this type II region, we can compute the required double integral as follows :

$$\iint\limits_D y\ dA = \int_0^2 \int_y^{4-y} y\ dx\ dy\ = \int_0^2 \left[xy\right]_{x=y}^{x=4-y}\ dy$$

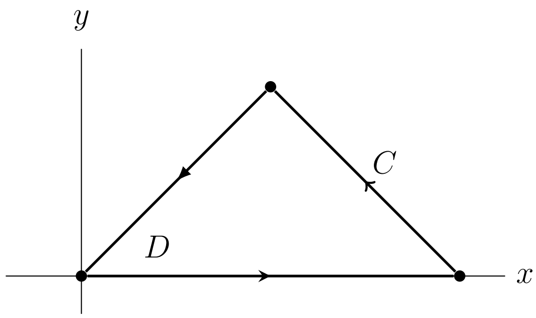
$$= \int_0^2 \left(\ y(4-y) - y^2\ \right) dy = \int_0^2 (4y - 2y^2)\ dy$$

$$= \left[2y^2 - \tfrac{2}{3}y^3\right]_{y=0}^{y=2}$$

$$= \tfrac{8}{3}.$$

(b)
The integral in (a) can represent a solid which lies above D and is bounded above by the plane $z = y$.

(c)



Since the conditions for Green's Theorem are satisfied, we can use the answer in (a) to compute the line integral around C, the boundary of D, as follows.

$$\int_C x^2\ dx + xy\ dy = \int_C\ P\ dx + Q\ dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

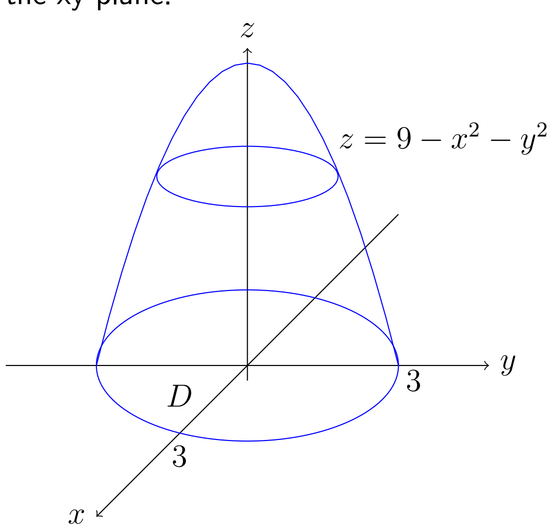
$$= \iint_D \left(\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(x^2)\right) dA = \iint_D y\ dA = \tfrac{8}{3}.$$

(d)

By making the identification $\mathbf{F} = x^2\mathbf{i} + xy\mathbf{j} = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$, we can express the line integral over $-C$ in terms of the line integral over C in part (c), enabling us to compute

$$\int_{-C} \mathbf{F}\cdot d\mathbf{r} = -\int_C \mathbf{F}\cdot d\mathbf{r} = -\int_C x^2\ dx + xy\ dy = -\tfrac{8}{3}.$$

6. Find the area of the part of the paraboloid $z = 9 - x^2 - y^2$ that lies above the xy-plane.



To determine the boundary of the domain D for this part of the paraboloid, set $9 - x^2 - y^2 = 0$, which is the circle $x^2 + y^2 = 9$. In rectangular and polar coordinates, we may respectively represent D as

$$D = \{(x, y) \mid x^2 + y^2 \leq 9\} = \{(r, \theta) \mid 0 \leq r \leq 3\ ,\ 0 \leq \theta \leq 2\pi\}.$$

Denote the paraboloid as $g(x, y) = 9 - x^2 - y^2$, and compute the surface area by using the appropriate formula as follows by switching to polar coordinates and using the separability property since the resulting integrand is separable (as a function of r only).

$$\begin{aligned} SA &= \iint\limits_D \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}\, dA = \iint\limits_D \sqrt{1 + 4(x^2 + y^2)}\, dA \\ &= \int_0^{2\pi} \int_0^3 \sqrt{1 + 4r^2} \cdot r\ dr d\theta = \int_0^{2\pi} d\theta \int_0^3 \sqrt{1 + 4r^2} \cdot r\ dr \end{aligned} \tag{3}$$

To evaluate the integral in r, use the substitution $u = 4r^2 + 1$, which implies that $\frac{1}{8} du = r dr$, $u(0) = 1$, and $u(3) = 37$. We then obtain that

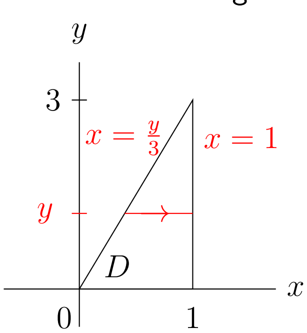
$$\begin{aligned} \int_0^3 \sqrt{1 + 4r^2} \cdot r\ dr = \tfrac{1}{8} \int_1^{37} \sqrt{u}\ du &= \left[\tfrac{1}{8} \cdot \tfrac{2}{3} u^{\frac{3}{2}}\right]_{u=1}^{u=37} \\ &= \tfrac{1}{12}\left(37\sqrt{37} - 1\right). \end{aligned}$$

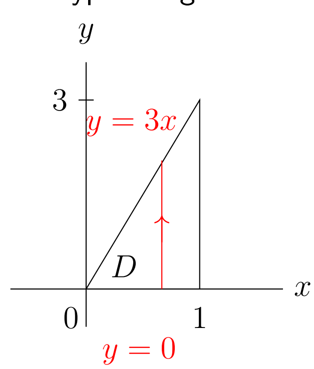
By direct computation, $\displaystyle\int_0^{2\pi} d\theta = 2\pi$. Upon substituting back into (6) we obtain that

$$SA = 2\pi \cdot \tfrac{1}{12}\left(37\sqrt{37} - 1\right) = \tfrac{\pi}{6}\left(37\sqrt{37} - 1\right).$$

7. Let E be the solid which is under the parabolic cylinder $z = x^2$ and above the region $D = \{(x, y, 0) \mid 0 \leq y \leq 3,\ \frac{1}{3} y \leq x \leq 1\}$. Do the following.
   (a) Simplify $\displaystyle\iiint\limits_E \cos z\ dV$ into an iterated double integral.
   (b) Switch the order of the integration in the double integral obtained in part (a), and simplify it into a single integral with limits. Do not evaluate.

(a) To switch the order of integration in the double integral in (b), we need to reconsider the given type I region D as a type II region.





As a type II region, $D = \{(x, y) \mid 0 \leq x \leq 1\ ,\ 0 \leq y \leq 3x\}$.

We can thus simplify the given triple integral into a single integral as required in parts (a) and (b) :

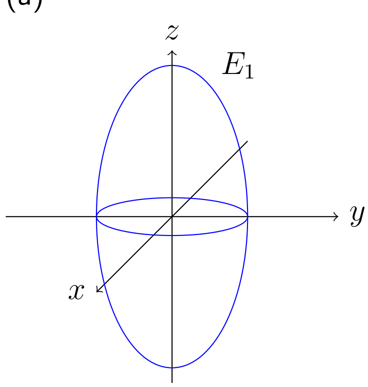
$$\iiint\limits_{D} \cos z \; dV = \iint\limits_{D} \int\limits_{0}^{x^2} \cos z \; dz \; dA = \iint_D \left[\sin z \right]_{z=0}^{z=x^2} \; dA$$
$$= \int\limits_{0}^{3} \int_{\frac{y}{3}}^{1} \sin x^2 \; dxdy = \int\limits_{0}^{1} \int\limits_{0}^{3x} \sin x^2 \; dydx$$
$$= \int\limits_{0}^{1} \left[y \sin x^2 \right]_{y=0}^{y=3x} \; dx = \int\limits_{0}^{1} 3x \sin x^2 \; dx.$$

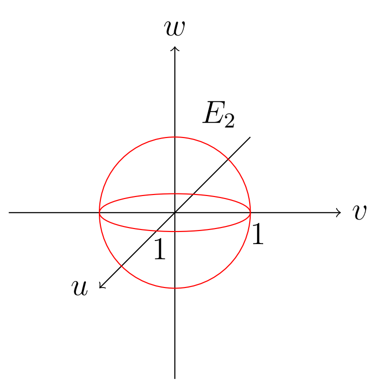
8. Let E_1 be the solid bounded by the ellipsoid $\frac{x^2}{9} + \frac{y^2}{4} + \frac{z^2}{25} = 1$. Do the following.
   (a) Show that the substitution $x = 3u$, $y = 2v$, and $z = 5w$ transforms E_1 into a solid E_2 which is bounded by the unit sphere centered at the origin.
   (b) Compute the Jacobian $\frac{\partial(x,y,z)}{\partial(u,v,w)}$ of this change in variables.
   (c) Use the Change in Variables Theorem to rewrite the following triple integral over E_1 as a triple integral over E_2 in terms of u, v, and w.

   $$\iiint\limits_{E_1} \sqrt{\frac{x^2}{9} + \frac{y^2}{4} + \frac{z^2}{25}} \, dV$$

   (d) Rewrite the triple integral over E_2 obtained in part (c) as a product of three single integrals with limits by first changing the integral over E_2 to spherical coordinates. Do not evaluate.

(a)





By direct substitution, the ellipsoid becomes $\frac{(3u)^2}{9} + \frac{(2v)^2}{4} + \frac{(5w)^2}{25} = 1$, which simplifies to $u^2 + v^2 + w^2 = 1$.

(b)

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{vmatrix} = 3 \cdot 2 \cdot 5 = 30.$$

(c), (d)
Note that in terms of spherical coordinates,
$E_2 = \{(\rho, \theta, \phi) \mid 0 \le \rho \le 1 \ , \ 0 \le \theta \le 2\pi \ , \ 0 \le \phi \le \pi\}$.
By applying the Change in Variables Theorem twice; first to change the domain of the original integral from E_1 to E_2, and second to rewrite the integral over E_2 using spherical coordinates, we obtain the following.

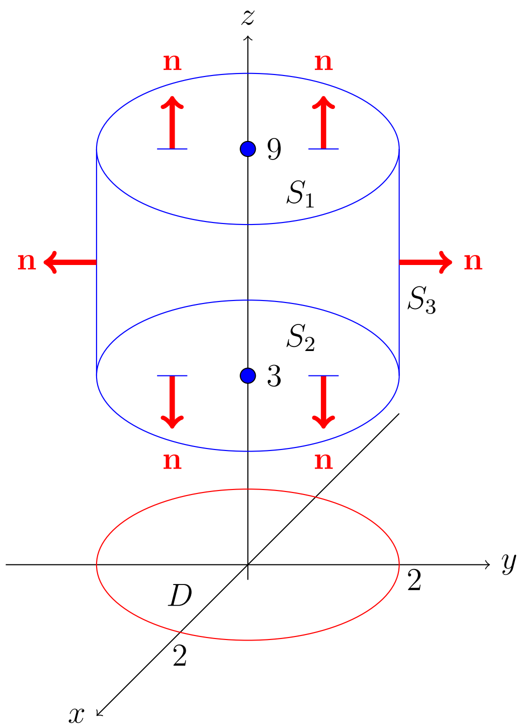
$$\begin{aligned} \iiint\limits_{E_1} \sqrt{\tfrac{x^2}{9} + \tfrac{y^2}{4} + \tfrac{z^2}{25}} \, dV &= \iiint\limits_{E_2} \sqrt{\tfrac{(3u)^2}{9} + \tfrac{(2v)^2}{4} + \tfrac{(5w)^2}{25}} \cdot \left| \tfrac{\partial(x,y,z)}{\partial(u,v,w)} \right| dV \\ &= \iiint\limits_{E_2} \sqrt{u^2 + v^2 + w^2} \cdot 30 \, dV \\ &= 30 \int_0^1 \int_0^{2\pi} \int_0^{\pi} \sqrt{\rho^2} \cdot \rho^2 \sin\phi \, d\phi \, d\theta \, d\rho. \\ &= 30 \int_0^1 \rho^3 d\rho \int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi \, d\phi. \end{aligned}$$

9. Let E be the right circular cylinder bounded by the surfaces $x^2 + y^2 = 4$, $z = 3$, and $z = 9$. The velocity field $\mathbf{F} = 3x\mathbf{i} + 2y\mathbf{j} + 4z\mathbf{k}$ describes how a liquid flows outward through the permeable surface S of E equipped with an outward unit normal vector function $\mathbf{n}$. Distance is measured in ft and time in min. Do the following.

   (a) Use Gauss's Divergence Theorem to evaluate $\iint\limits_S \mathbf{F} \cdot \mathbf{n} \, dS$.

   (b) Let S_1 and S_2 respectively be the top and bottom circular disks of E. Evaluate $\iint\limits_{S_1} \mathbf{F} \cdot \mathbf{n} \, dS$ and $\iint\limits_{S_2} \mathbf{F} \cdot \mathbf{n} \, dS$.

   (c) Use the results in part (a) and (b) to find the flux of $\mathbf{F}$ through S_3, the part of S bounded by $x^2 + y^2 = 4$. Indicate units of measure, and explain what your answer means in terms of liquid flow.



(a)

$$
\begin{aligned}
\iint_S \mathbf{F}\cdot\mathbf{n}\, dS &= \iiint_E \nabla\cdot\mathbf{F}\, dV \\
&= \iiint_E \left(\frac{\partial}{\partial x}(3x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(4z)\right) dV \\
&= \iiint_E 9\ dV = 9 \iiint_E 1\ dV = 9\cdot \text{Volume}(E) \\
&= 9\cdot \pi(2)^2 \cdot 6 \\
&= 216\pi.
\end{aligned}
$$

(b)

The domain for S, S_1, and S_2 is D, which is a circle centered at the origin with a radius of 2. The surface S_1 has equation $g_1(x,y) = 9$, for which $\frac{\partial g_1}{\partial x} = 0$ and $\frac{\partial g_1}{\partial y} = 0$. The pre-unit normal vector to S_1 is given by

$$\mathbf{N} = -\frac{\partial g_1}{\partial x}\mathbf{i} - \frac{\partial g_1}{\partial y}\mathbf{j} + \mathbf{k} = 0\mathbf{i} + 0\mathbf{j} + 1\mathbf{k}.$$

Therefore,

$$\begin{aligned}
\iint\limits_{S_1} \mathbf{F}\cdot\mathbf{n}\,dS &= \iint\limits_{D} \mathbf{F}\cdot\mathbf{N}\,dA \\
&= \iint\limits_{S_1} (3x\mathbf{i}+2y\mathbf{j}+4z\mathbf{k})\cdot(0\mathbf{i}+0\mathbf{j}+1\mathbf{k})dA \\
&= \iint\limits_{D} 4z\ dA = \iint_D 4\cdot 9\ dA = 36\iint_D 1\ dA \\
&= 36\cdot \text{Area}(D) = 36\pi\cdot 2^2 = 144\pi.
\end{aligned}$$

Likewise, the surface S_2 has equation $g_2(x,y)=3$, for which $\frac{\partial g_2}{\partial x}=0$ and $\frac{\partial g_2}{\partial y}=0$. The pre-unit normal vector to S_2 is given by

$$\mathbf{N}=\frac{\partial g_1}{\partial x}\mathbf{i}+\frac{\partial g_1}{\partial y}\mathbf{j}-\mathbf{k}=0\mathbf{i}+0\mathbf{j}-1\mathbf{k}.$$

Therefore,

$$\begin{aligned}
\iint\limits_{S_1} \mathbf{F}\cdot\mathbf{n}\,dS &= \iint\limits_{D} \mathbf{F}\cdot\mathbf{N}\,dA \\
&= \iint\limits_{S_2} (3x\mathbf{i}+2y\mathbf{j}+4z\mathbf{k})\cdot(0\mathbf{i}+0\mathbf{j}-1\mathbf{k})dA \\
&= \iint\limits_{D} -4z\ dA = \iint_D -4\cdot 3\ dA = 36\iint_D 1\ dA \\
&= -12\cdot \text{Area}(D) = -12\pi\cdot 2^2 = -48\pi.
\end{aligned}$$

(c)
By the additivity property for surface integrals,

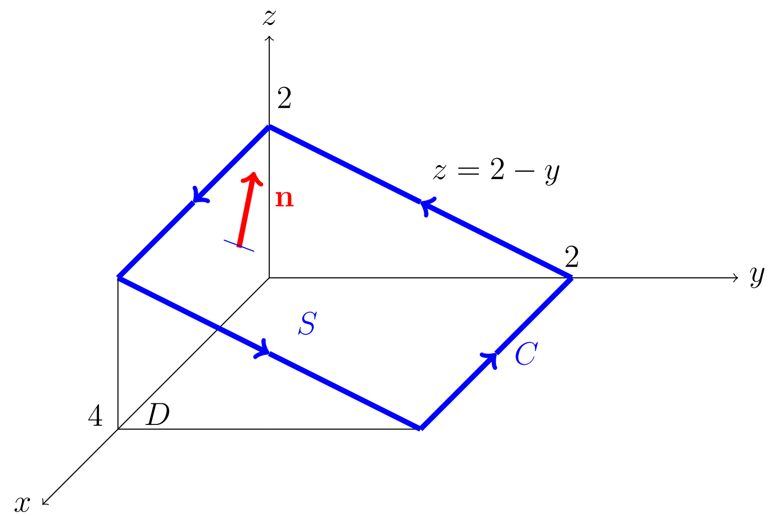
$$\begin{aligned}
\iint_{S_1}\mathbf{F}\cdot\mathbf{n}\,dS \ + \ \iint_{S_2}\mathbf{F}\cdot\mathbf{n}\,dS \ + \ \iint_{S_3}\mathbf{F}\cdot\mathbf{n}\,dS \ &= \iint_S \mathbf{F}\cdot\mathbf{n}\,dS \\
144\pi - 48\pi + \iint\limits_{S_3}\mathbf{F}\cdot\mathbf{n}\,dS &= 216\pi \\
\iint_{S_3}\mathbf{F}\cdot\mathbf{n}\,dS &= 120\pi.
\end{aligned}$$

The flux through S_3 is $120\frac{\text{ft}^3}{\text{min}}$, which means that in 1 minute 120π ft^3 of liquid permeates through S_3.

10. Let S be the rectangle which is the part of the plane $y + z = 2$ that has vertices at $(4, 2, 0), (0, 2, 0), (0, 0, 2)$ and $(4, 0, 2)$ with boundary C. Let the force field $\mathbf{F} = 2y\mathbf{i} + 8x\mathbf{j} - 5xz\mathbf{k}$. Do the following.

   (a) Show that $\nabla \times \mathbf{F} = 5z\mathbf{j} + 6\mathbf{k}$.

   (b) State the equation for Stokes' Theorem and use it to set up a simplified, iterated double integral that equals the work done by $\mathbf{F}$ in moving a particle counter-clockwise around C. Do not evaluate.



(a)
For the force field $\mathbf{F} = 2y\mathbf{i} + 8x\mathbf{j} - 5xz\mathbf{k}$, we have that

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y & 8x & -5xz \end{vmatrix}$$

$$= \mathbf{i}\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 8x & -5xz \end{vmatrix} - \mathbf{j}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ 2y & -5xz \end{vmatrix} + \mathbf{k}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ 2y & 8x \end{vmatrix}$$

$$= \mathbf{i}(0 - 0) - \mathbf{j}(-5z - 0) + \mathbf{k}(8 - 2)$$

$$= 0\mathbf{i} + 5z\mathbf{j} + 6\mathbf{k}.$$

(b)
Denote the equation for the surface S by $g(x, y) = 2 - y$. Note that the domain D of S is given by $D = \{(x, y) \mid 0 \le x \le 2\ ,\ 0 \le y \le 4\}$. By the right hand thumb rule, since C is oriented counter-clockwise, the unit normal vector $\mathbf{n}$ to S has an upward $\mathbf{k}$ component. Therefore, the associated pre-unit normal vector to S is

$$\mathbf{N} = -\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k} = 0\mathbf{i} + 1\mathbf{j} + 1\mathbf{k}.$$

Therefore, by applying Stokes' Theorem, the work done by $\mathbf{F}$ in moving a particle counter-clockwise is

$$\begin{aligned}
\text{Work} &= \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \big(\nabla \times \mathbf{F}\big) \cdot \mathbf{n}\ dS = \iint_D \big(\nabla \times \mathbf{F}\big) \cdot \mathbf{N}\ dA \\
&= \int_0^4 \int_0^2 \big(0\mathbf{i} + 5z\mathbf{j} + 6\mathbf{k}\big) \cdot \big(0\mathbf{i} + 1\mathbf{j} + 1\mathbf{k}\big) dy\ dx = \int_0^4 \int_0^2 \big(5z + 6\big) dy\ dx \\
&= \int_0^4 \int_0^2 \big(5(2 - y) + 6\big)\ dxdy = \int_0^4 \int_0^2 \big(16 - 5y\big) dydx.
\end{aligned}$$

Calculus III Final Exam Form B Solution Key
Fall 2025

1. A particle traveling in a circle has velocity function $v(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j}$ with initial position $\mathbf{r}(0) = 2\mathbf{j} + 2\mathbf{k}$.

   (a) Find the position function $\mathbf{r}(t)$ for this particle.

   (b) Find the distance traveled by this particle over $0 \leq t \leq \frac{\pi}{2}$.

(a) $\mathbf{r}'(t) = 3\sin t\mathbf{i} + 3\cos t\mathbf{j}$, $\mathbf{r}(0) = 3\mathbf{j} - 3\mathbf{k}$.

$$\begin{aligned}
\int \mathbf{r}'(t)dt &= \int (2\cos t\mathbf{i} + 2\sin t\mathbf{j})dt \\
\mathbf{r}(t) &= 2\sin t\mathbf{i} - 2\cos t\mathbf{j} + \mathbf{c} \\
\mathbf{r}(0) &= 2\sin(0)\mathbf{i} - 2\cos(0)\mathbf{j} + \mathbf{c} \\
2\mathbf{j} + 2\mathbf{k} &= 0\mathbf{i} - 2\mathbf{j} + \mathbf{c} \\
4\mathbf{j} + 2\mathbf{k} &= \mathbf{c}.
\end{aligned}$$

Therefore, $\mathbf{r}(t) = 2\sin t\mathbf{i} + (-2\cos t + 4)\mathbf{j} + 2\mathbf{k}$.

(b)

$$\begin{aligned}
TD &= \int_0^{\frac{\pi}{2}} \left|\mathbf{v}(t)\right| dt = \int_0^{\frac{\pi}{2}} \sqrt{(2\cos t)^2 + (2\sin t)^2}\ dt \\
&= \int_0^{\frac{\pi}{2}} \sqrt{4(\cos^2 t + \sin^2 t)}dt = \int_0^{\frac{\pi}{2}} 2\ dt = \left[2t\right]_0^{\frac{\pi}{2}} = \pi.
\end{aligned}$$

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2. Do the following, where $z = f(x, y)$ where $x = r\cos 3\theta$, $y = r\sin 3\theta$.

   (a) Using limit notation, write what it means for the function f_x to be continuous at the point (a, b).

   (b) Find $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$ if f_x and f_y are continuous.

   (c) Show that $y = r\sin 2\theta$ is a solution to the partial differential equation $9(x_r)^2 + (x_{\theta r})^2 + 9(x_{r\theta})^2 + (x_{r\theta\theta})^2 = 90$.

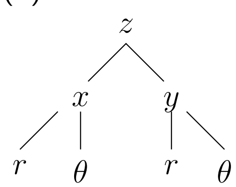
(a) The function f_x is continuous at the point (a, b) means that:

(i) $f_x(a, b)$ exists

(ii) $\lim\limits_{(x,y)\to(a,b)} f_x(x, y)$ exists

(iii) $\lim\limits_{(x,y)\to(a,b)} f_x(x, y) = f_y(a, b)$.

(b)



$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r}$$
$$= f_x(x,y)\cos 3\theta \ + \ f_y(x,y)\sin 3\theta.$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$
$$= -3f_x(x,y)r\sin 3\theta \ +3 \ f_y(x,y)r\cos 3\theta.$$

(c)

$x_r = \cos 3\theta$, $x_{r\theta} = -3\sin 3\theta$, $x_{\theta r} = -3\sin 3\theta$, $x_{r\theta\theta} = -9\cos 3\theta$

$$\begin{aligned} 9(x_r)^2 + (x_{\theta r})^2 + 9(x_{r\theta})^2 + (x_{r\theta\theta})^2 &= 9\cos^2 3\theta + 9\cdot 9sin^2 3\theta + 9\sin^2 3\theta + 81\cos^2 2\theta \\ &= 9(\sin^2 3\theta + \cos^2 3\theta) + 81(\sin^2 3\theta + \cos^2 3\theta) \\ &= 9\cdot 1 + 81\cdot 1 = 90. \end{aligned}$$

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3. Let $f(x,y) = xy + e^{2x+y}$ denote the temperature of a circular disk in °F. Assume distance is measured in feet.
   (a) Find the equation of the tangent plane to $z = f(x,y)$ at $(-1,2)$.
   (b) Find $D_u f(-1,2)$ in the direction of $(3,5)$, indicate units of measure, and explain what $D_u f(-1,2)$ means.

(a)

$f_x = y + 2e^{2x+y}$, $f_y = x + e^{2x+y}$; so therefore
$f(-1,2) = -1,\ f_x(-1,2) = 4,\ f_y(-1,2) = 0.$ The equation of the tangent plane is therefore

$$\begin{aligned} z &= f(-1,2) + f_x(-1,2)(x+1) + f_y(-1,2)(y-2) \\ &= -1 + 4(x+1) + 0(y-2) \\ z &= 4x + 3. \end{aligned}$$

(b)

$\mathbf{v} = \langle 3,5\rangle - \langle -1,2\rangle = \langle 4,3\rangle$. $|\mathbf{v}| = \sqrt{4^2+3^2} = 5$ and $\nabla f(-1,2) = \langle 4,0\rangle$.
Therefore $\mathbf{u} = \langle \frac{4}{5}, \frac{3}{5}\rangle$ and so

$$D_u f(-1,2) = \nabla f(-1,2)\cdot\mathbf{u} = \langle 4,0\rangle\cdot\langle \tfrac{4}{5},\tfrac{3}{5}\rangle = \tfrac{16}{5}\,\tfrac{{}^\circ\mathrm{F}}{\mathrm{ft}}. \qquad (4)$$

The answer in (4) means that the temperature of the circular disk will increase by 16°F over a 5ft increase in the direction of $\mathbf{v}$ from the point $(-1,2)$.

4. The plane $z = 4x + 6y$ intersects the cylinder $x^2 + y^2 = 13$ in an ellipse. Find the maximum and minimum perpendicular distances from this ellipse to the xy-plane.

The goal is to optimize $f(x, y) = 4x + 6y$ subject to the constraint $g(x, y) = 13$ where $g(x, y) = x^2 + y^2$. By the method of Lagrange multipliers, we have that

$$\nabla f(x, y) = \lambda \nabla g(x, y)$$
$$f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j} = \lambda g_x(x, y)\mathbf{i} + \lambda g_y(x, y)\mathbf{j},$$

which implies that

$$\begin{aligned} f_x(x, y) &= \lambda g_x(x, y) & \qquad f_y(x, y) &= \lambda g_y(x, y) \\ 4 &= \lambda \cdot 2x & 6 &= \lambda \cdot 2y \\ \frac{2}{\lambda} &= x & \frac{3}{\lambda} &= y \end{aligned} \tag{5}$$

By substitution into $g(x, y) = 13$ we obtain that

$$\begin{aligned} x^2 + y^2 &= 13 \\ \frac{4}{\lambda^2} + \frac{9}{\lambda^2} &= 13 \\ \lambda^2 &= 1 \end{aligned}$$

By substitution into (5) the two possibilities $\lambda = 1$, $\lambda = -1$ respectively correspond to the points $(-1, -4)$ and $(1, 4)$ on the circle $x^2 + y^2 = 17$. Thus the maximum and minimum values of $f(x, y)$ subject to $g(x, y) = 17$ are $f(-2, -3) = -26$ and $f(2, 3) = 26$. This means that the points on the ellipse that are the greatest distance from the xy-plane are $(-2, -3, -26)$ and $(2, 3, 26)$, and so 26 is the maximum distance from the ellipse to the xy-plane.

Since $f(x, y)$ is continuous over the domain $\{(x, y) \mid g(x, y) = 13\}$ and $f(-2, -3) = -26 < 0 < 26 = f(2, 3)$ the Intermediate Value Theorem implies that there is at least one point on the ellipse where $f(x, y) = 0$; and hence the minimum distance from the ellipse to the xy-plane is zero. To find where this occurs, set

$$\begin{aligned} f(x, y) = 4x + 6y &= 0 \text{ , which implies that} \\ x &= -\tfrac{3}{2}y \text{ , and so by substitution into } g(x, y) = 13 \\ (-\tfrac{3}{2}y)^2 + y^2 &= 17 \;\Rightarrow \\ y &= \pm 2 \text{ , } x = \mp 3 \end{aligned}$$

Therefore, the minimum distance of zero from the ellipse to the xy-plane occurs at the two points $(-3, 2, 0)$ and $(3, -2, 0)$.

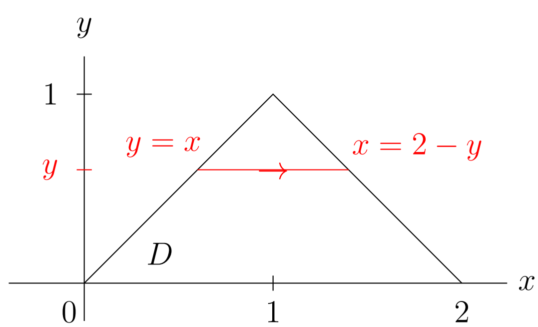
5. Let D be the triangular region in the xy-plane with vertices at $(0,0,0),(1,1,0)$, and $(2,0,0)$, which is bounded by $y=x$, $x+y=2$ and $y=0$. Do the following.

(a) Show that $\displaystyle\iint_D y\ dA = \frac{1}{3}$ by evaluating this integral.

(b) Describe a solid which the integral in (a) represents the volume of.

(c) If C is the boundary of D oriented counter-clockwise, use Green's Theorem and your answer in part (a) to evaluate the line integral $\displaystyle\int_C x\ dx\ +\ xy\ dy.$

(d) Let $-C$ be the traversal of C in a clockwise direction. Use your answer in (c) to set up and state the value of a line integral over $-C$ that equals the work done by $\mathbf{F} = x^2\mathbf{i} + xy\mathbf{j}$ in moving a particle along $-C$ (in a clock-wise direction).

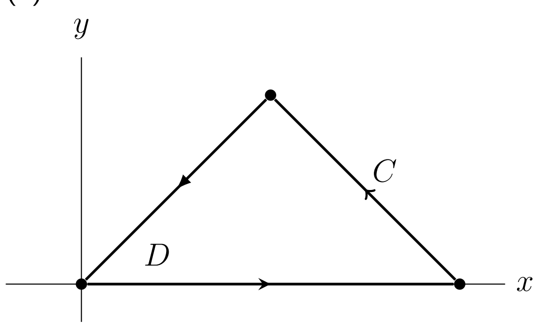


(a)
As a type II region, we can express the domain of integration as $D = \{(x,y) \mid 0 \le y \le 1\ ,\ y \le x \le 2-y\}$. Using the limits prescribed by this type II region, we can compute the required double integral as follows :

$$\iint_D y\ dA = \int_0^1 \int_y^{2-y} y\ dx\ dy\ = \int_0^1 [xy]_{x=y}^{x=2-y}\ dy$$
$$= \int_0^1 (\ y(2-y) - y^2\ )dy = \int_0^1 (2y - 2y^2)\ dy$$
$$= \left[y^2 - \tfrac{2}{3}y^3\right]_{y=0}^{y=1}$$
$$= \tfrac{1}{3}.$$

(b)
The integral in (a) can represent a solid which lies above D and is bounded above by the plane $z = y$.

(c)



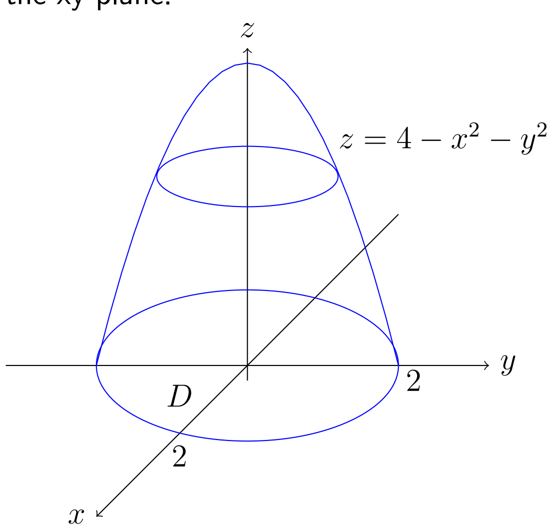
Since the conditions for Green's Theorem are satisfied, we can use the answer in (a) to compute the line integral around C, the boundary of D, as follows.

$$\int_C x\ dx + xy\ dy = \int_C P\ dx + Q\ dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$
$$= \iint_D \left(\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(x)\right) dA = \iint_D y\ dA = \tfrac{1}{3}.$$

(d)
By making the identification $\mathbf{F} = x\mathbf{i} + xy\mathbf{j} = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$, we can express the line integral over $-C$ in terms of the line integral over C in part (c), enabling us to compute

$$\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int_C \mathbf{F} \cdot d\mathbf{r} = -\int_C x\ dx + xy\ dy = -\tfrac{1}{3}.$$

6. Find the area of the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the xy-plane.



To determine the boundary of the domain D for this part of the paraboloid, set $4 - x^2 - y^2 = 0$, which is the circle $x^2 + y^2 = 4$. In rectangular and polar coordinates, we may respectively represent D as

$$D = \{(x,y) \mid x^2+y^2 \le 4\} = \{(r,\theta) \mid 0 \le r \le 2\ ,\ 0 \le \theta \le 2\pi\}.$$

Denote the paraboloid as $g(x,y) = 4 - x^2 - y^2$, and compute the surface area by using the appropriate formula as follows by switching to polar coordinates and using the separability property since the resulting integrand is separable (as a function of r only).

$$\begin{aligned} SA &= \iint_D \sqrt{1+\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2}\, dA = \iint_D \sqrt{1+4(x^2+y^2)}\, dA \\ &= \int_0^{2\pi}\int_0^2 \sqrt{1+4r^2}\cdot r\, dr d\theta = \int_0^{2\pi} d\theta \int_0^2 \sqrt{1+4r^2}\cdot r\, dr \end{aligned} \tag{6}$$

To evaluate the integral in r, use the substitution $u = 4r^2+1$, which implies that $\frac{1}{8}du = rdr$, $u(0) = 1$, and $u(3) = 17$. We then obtain that

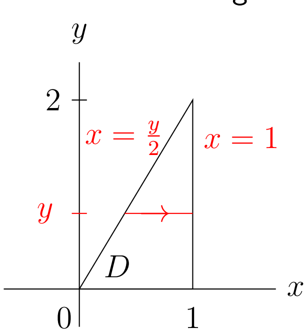
$$\begin{aligned} \int_0^2 \sqrt{1+4r^2}\cdot r\, dr = \tfrac{1}{8}\int_1^{17}\sqrt{u}\, du &= \left[\tfrac{1}{8}\cdot\tfrac{2}{3}u^{\frac{3}{2}}\right]_{u=1}^{u=17} \\ &= \tfrac{1}{12}\left(17\sqrt{17}-1\right). \end{aligned}$$

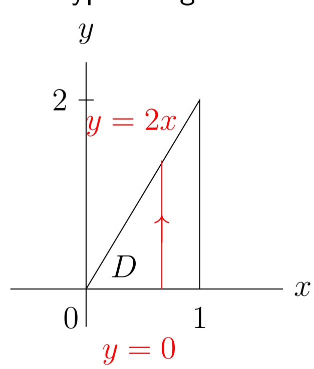
By direct computation, $\displaystyle\int_0^{2\pi} d\theta = 2\pi$. Upon substituting back into (6) we obtain that

$$SA = 2\pi\cdot\tfrac{1}{12}\left(17\sqrt{17}-1\right) = \tfrac{\pi}{6}\left(17\sqrt{17}-1\right).$$

7. Let E be the solid which is under the parabolic cylinder $z = x^2$ and above the region $D = \{(x,y,0) \mid 0 \le y \le 2,\ \frac{1}{2}y \le x \le 1\}$. Do the following.
   (a) Simplify $\displaystyle\iiint_E \cos z\, dV$ into an iterated double integral.
   (b) Switch the order of the integration in the double integral obtained in part (a), and simplify it into a single integral with limits. Do not evaluate.

(a) To switch the order of integration in the double integral in (b), we need to reconsider the given type I region D as a type II region.





As a type II region, $D = \{(x,y) \mid 0 \le x \le 1\ ,\ 0 \le y \le 2x\}$.

We can thus simplify the given triple integral into a single integral as required in parts (a) and (b) :

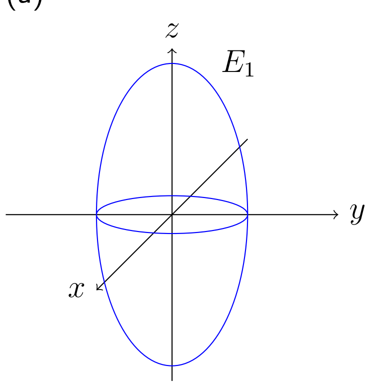
$$\iiint\limits_D \cos z \; dV = \iint\limits_D \int\limits_0^{x^2} \cos z \; dz \; dA = \iint_D \left[\sin z \right]_{z=0}^{z=x^2} \; dA$$
$$= \int\limits_0^2 \int_{\frac{y}{2}}^1 \sin x^2 \; dxdy = \int\limits_0^1 \int\limits_0^{2x} \sin x^2 \; dydx$$
$$= \int\limits_0^1 \left[y \sin x^2 \right]_{y=0}^{y=2x} \; dx = \int\limits_0^1 2x \sin x^2 \; dx.$$

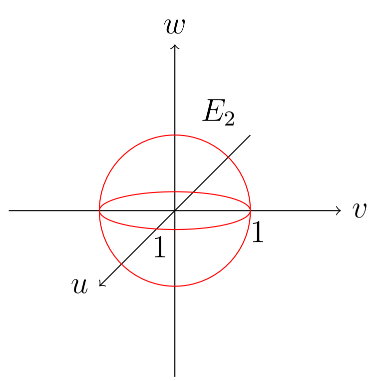
8. Let E_1 be the solid bounded by the ellipsoid $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1$. Do the following.
   (a) Show that the substitution $x = 2u$, $y = 3v$, and $z = 4w$ transforms E_1 into a solid E_2 which is bounded by the unit sphere centered at the origin.
   (b) Compute the Jacobian $\frac{\partial(x,y,z)}{\partial(u,v,w)}$ of this change in variables.
   (c) Use the Change in Variables Theorem to rewrite the following triple integral over E_1 as a triple integral over E_2 in terms of u, v, and w.

   $$\iiint\limits_{E_1} \sqrt{\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16}} \, dV$$

   (d) Rewrite the triple integral over E_2 obtained in part (c) as a product of three single integrals with limits by first changing the integral over E_2 to spherical coordinates. Do not evaluate.

(a)





By direct substitution, the ellipsoid becomes $\frac{(2u)^2}{4} + \frac{(3v)^2}{9} + \frac{(4w)^2}{16} = 1$, which simplifies to $u^2 + v^2 + w^2 = 1$.

(b)

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{vmatrix} = 2 \cdot 3 \cdot 4 = 24.$$

(c), (d)
Note that in terms of spherical coordinates,
$E_2 = \{(\rho, \theta, \phi) \mid 0 \le \rho \le 1\ ,\ 0 \le \theta \le 2\pi\ ,\ 0 \le \phi \le \pi\}$.
By applying the Change in Variables Theorem twice; first to change the domain of the original integral from E_1 to E_2, and second to rewrite the integral over E_2 using spherical coordinates, we obtain the following.

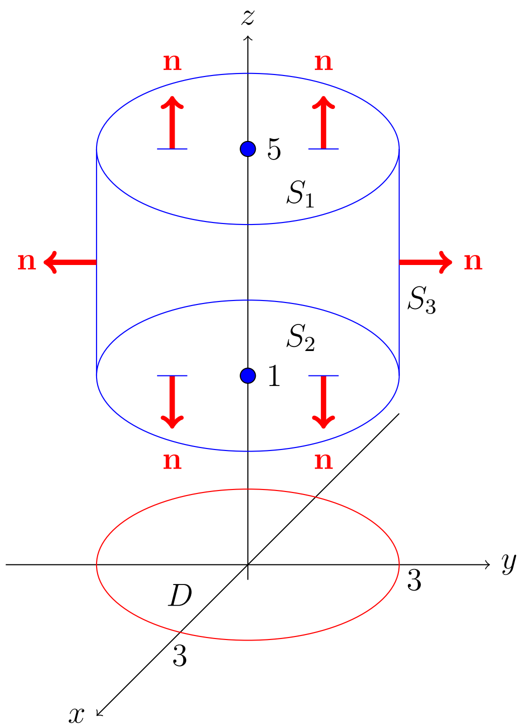
$$\begin{aligned} \iiint_{E_1} \sqrt{\tfrac{x^2}{4} + \tfrac{y^2}{9} + \tfrac{z^2}{16}}\ dV &= \iiint_{E_2} \sqrt{\tfrac{(3u)^2}{9} + \tfrac{(2v)^2}{4} + \tfrac{(5w)^2}{25}} \cdot \left|\tfrac{\partial(x,y,z)}{\partial(u,v,w)}\right| dV \\ &= \iiint_{E_2} \sqrt{u^2 + v^2 + w^2} \cdot 24\ dV \\ &= 24 \int_0^1 \int_0^{2\pi} \int_0^{\pi} \sqrt{\rho^2} \cdot \rho^2 \sin\phi\ d\phi\ d\theta\ d\rho. \\ &= 24 \int_0^1 \rho^3 d\rho \int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi\ d\phi. \end{aligned}$$

9. Let E be the right circular cylinder bounded by the surfaces $x^2 + y^2 = 9$, $z = 1$, and $z = 5$. The velocity field $\mathbf{F} = 2x\mathbf{i} + 3y\mathbf{j} + 4z\mathbf{k}$ describes how a liquid flows outward through the permeable surface S of E equipped with an outward unit normal vector function $\mathbf{n}$. Distance is measured in ft and time in min. Do the following.

   (a) Use Gauss's Divergence Theorem to evaluate $\iint_S \mathbf{F} \cdot \mathbf{n}\ dS$.

   (b) Let S_1 and S_2 respectively be the top and bottom circular disks of E. Evaluate $\iint_{S_1} \mathbf{F} \cdot \mathbf{n}\ dS$ and $\iint_{S_2} \mathbf{F} \cdot \mathbf{n}\ dS$.

   (c) Use the results in part (a) and (b) to find the flux of $\mathbf{F}$ through S_3, the part of S bounded by $x^2 + y^2 = 9$. Indicate units of measure, and explain what your answer means in terms of liquid flow.



(a)

$$\begin{aligned}
\iint_S \mathbf{F} \cdot \mathbf{n}\, dS &= \iiint_E \nabla \cdot \mathbf{F}\, dV \\
&= \iiint_E \left(\frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(3y) + \frac{\partial}{\partial z}(4z) \right) dV \\
&= \iiint_E 9\, dV = 9 \iiint_E 1\, dV = 9 \cdot \text{Volume}(E) \\
&= 9 \cdot \pi(3)^2 \cdot 4 \\
&= 324\pi.
\end{aligned}$$

(b)
The domain for S, S_1, and S_2 is D, which is a circle centered at the origin with a radius of 3. The surface S_1 has equation $g_1(x, y) = 5$, for which $\frac{\partial g_1}{\partial x} = 0$ and $\frac{\partial g_1}{\partial y} = 0$. The pre-unit normal vector to S_1 is given by

$$\mathbf{N} = -\frac{\partial g_1}{\partial x}\mathbf{i} - \frac{\partial g_1}{\partial y}\mathbf{j} + \mathbf{k} = 0\mathbf{i} + 0\mathbf{j} + 1\mathbf{k}.$$

Therefore,

$$\begin{aligned}
\iint\limits_{S_1} \mathbf{F}\cdot\mathbf{n}\, dS &= \iint\limits_{D} \mathbf{F}\cdot\mathbf{N}\, dA \\
&= \iint\limits_{S_1} (2x\mathbf{i}+3y\mathbf{j}+4z\mathbf{k})\cdot(0\mathbf{i}+0\mathbf{j}+1\mathbf{k})dA \\
&= \iint\limits_{D} 4z\ dA = \iint_D 4\cdot 5\ dA = 20\iint_D 1\ dA \\
&= 36\cdot \text{Area}(D) = 20\pi\cdot 3^2 = 180\pi.
\end{aligned}$$

Likewise, the surface S_2 has equation $g_2(x,y)=1$, for which $\frac{\partial g_2}{\partial x}=0$ and $\frac{\partial g_2}{\partial y}=0$. The pre-unit normal vector to S_2 is given by

$$\mathbf{N} = \frac{\partial g_1}{\partial x}\mathbf{i} + \frac{\partial g_1}{\partial y}\mathbf{j} - \mathbf{k} = 0\mathbf{i}+0\mathbf{j}-1\mathbf{k}.$$

Therefore,

$$\begin{aligned}
\iint\limits_{S_1} \mathbf{F}\cdot\mathbf{n}\, dS &= \iint\limits_{D} \mathbf{F}\cdot\mathbf{N}\, dA \\
&= \iint\limits_{S_2} (2x\mathbf{i}+3y\mathbf{j}+4z\mathbf{k})\cdot(0\mathbf{i}+0\mathbf{j}-1\mathbf{k})dA \\
&= \iint\limits_{D} -4z\ dA = \iint_D -4\cdot 1\ dA = 36\iint_D 1\ dA \\
&= -4\cdot \text{Area}(D) = -4\pi\cdot 3^2 = -36\pi.
\end{aligned}$$

(c)
By the additivity property for surface integrals,

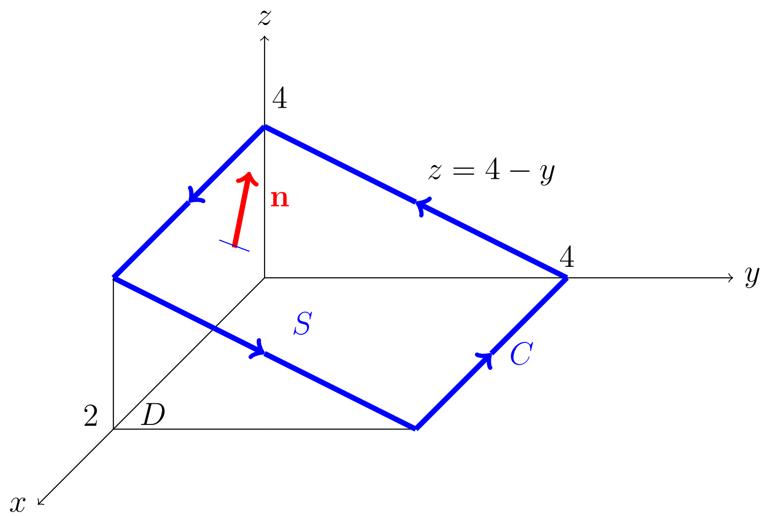
$$\begin{aligned}
\iint_{S_1} \mathbf{F}\cdot\mathbf{n}\, dS \ + \ \iint_{S_2} \mathbf{F}\cdot\mathbf{n}\, dS \ + \ \iint_{S_3} \mathbf{F}\cdot\mathbf{n}\, dS \ &= \iint_{S} \mathbf{F}\cdot\mathbf{n}\, dS \\
180\pi - 36\pi + \iint\limits_{S_3} \mathbf{F}\cdot\mathbf{n}\, dS &= 324\pi \\
\iint_{S_3} \mathbf{F}\cdot\mathbf{n}\, dS &= 180\pi.
\end{aligned}$$

The flux through S_3 is $180\frac{\text{ft}^3}{\text{min}}$, which means that in 1 minute 180π ft^3 of liquid permeates through S_3.

10. Let S be the rectangle which is the part of the plane $y + z = 4$ that has vertices at $(2, 4, 0), (0, 4, 0), (0, 0, 4)$ and $(2, 0, 4)$ with boundary C. Let the force field $\mathbf{F} = 4y\mathbf{i} + 6x\mathbf{j} - 3xz\mathbf{k}$. Do the following.

   (a) Show that $\nabla \times \mathbf{F} = 3z\mathbf{j} + 2\mathbf{k}$.

   (b) State the equation for Stokes' Theorem and use it to set up a simplified, iterated double integral that equals the work done by $\mathbf{F}$ in moving a particle counter-clockwise around C. Do not evaluate.



(a)

For the force field $\mathbf{F} = 4y\mathbf{i} + 6x\mathbf{j} - 3xz\mathbf{k}$, we have that

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4y & 6x & -3xz \end{vmatrix}$$

$$= \mathbf{i} \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6x & -3xz \end{vmatrix} - \mathbf{j} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ 4y & -3xz \end{vmatrix} + \mathbf{k} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ 4y & 6x \end{vmatrix}$$

$$= \mathbf{i}(0 - 0) - \mathbf{j}(-3z - 0) + \mathbf{k}(6 - 4)$$

$$= 0\mathbf{i} + 3z\mathbf{j} + 2\mathbf{k}.$$

(b)

Denote the equation for the surface S by $g(x, y) = 4 - y$. Note that the domain D of S is given by $D = \{(x, y) \mid 0 \le x \le 4\ ,\ 0 \le y \le 2\}$. By the right hand thumb rule, since C is oriented counter-clockwise, the unit normal vector $\mathbf{n}$ to S has an upward $\mathbf{k}$ component. Therefore, the associated pre-unit normal vector to S is

$$\mathbf{N} = -\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k} = 0\mathbf{i} + 1\mathbf{j} + 1\mathbf{k}.$$

Therefore, by applying Stokes' Theorem, the work done by $\mathbf{F}$ in moving a particle counter-clockwise is

$$\text{Work} = \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\; dS = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{N}\; dA$$

$$= \int_0^4 \int_0^2 \big(0\mathbf{i} + 3z\mathbf{j} + 2\mathbf{k}\big) \cdot \big(0\mathbf{i} + 1\mathbf{j} + 1\mathbf{k}\big) dy\; dx = \int_0^2 \int_0^4 \big(3z + 2\big) dy\; dx$$

$$= \int_0^2 \int_0^4 \big(3(4-y) + 2\big)\; dx dy = \int_0^4 \int_0^2 \big(14 - 3y\big) dy dx.$$